Flatness of the Commutator Map on SL_n

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Outline

- **Definition** : for any $k \in \mathbb{Z}_+$ and any group G, $\phi_k = [,]^k : G^{2k} \to G$ via

$$\phi_k(a_1, b_1, a_2, b_2, \cdots, a_k, b_k) = [a_1, b_1] \cdots [a_k, b_k],$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$.

- **Question** : is ϕ_k *flat* for any *simple algebraic group* G and $k \geq 1$?

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- **Question** : is ϕ_k *flat* for any *simple algebraic group* G and $k \geq 1$?

- For G a (connected) algebraic group over $\mathbb{C}$, the fiber $\phi_k^{-1}(g), \forall g \in G$, is an algebraic set, hence inherits both the Zariski and Euclidean topology of G^{2k}. The the commutator map ϕ_k is flat $\leftrightarrow$ the fibers $\phi_k^{-1}(g)$ all have equal dimension.
- **Example** : For $G = GL_n(\mathbb{C})$,

$$\phi_k : GL_n(\mathbb{C})^{2k} \to GL_n(\mathbb{C}),$$

with image in $SL_n(\mathbb{C})$. For $k \geq 2$, the commutator map is flat and

$$\dim \phi_k^{-1}(g) = (2k-1)n^2 + 1, \forall g \in SL_n(\mathbb{C}).$$

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- For $k = 1$, $\phi_1 = [,] : GL_n \times GL_n \to SL_n$ is not flat :

Theorem (Larsen-L)

$$\dim[,]^{-1}(g) = n^2 + 1, \forall g \in SL_n(K) \ \textit{not scalar.}$$

For $ord(\xi) = m$ *in* $K^\times$,

$$\dim[,]^{-1}(\xi I_n) = n^2 + m.$$

Remarks

- The result is not just about linear algebra.
- For other families of groups of Lie type, the result is worse. For $G = B_2$ the symplectic group of rank 4, $\dim[,]^{-1}(g)$ are not all equal for g not scalar.

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For $k \geq 2$, $\phi_k = [,]^k : G^{2k} \to G$ is flat for any simple algebraic group G.

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It starts with a character formula by Frobenius

Theorem (Frobenius, 1896)

For G any finite group and $g \in G$, the number of solutions in G^{2k} to the equation $[x_1, y_1] \cdots [x_k, y_k] = g$ for any $k \geq 1$ equals

$$|G|^{2k-1} \sum_{\chi \in \mathrm{Irr(G)}} \frac{\chi(g)}{\chi(1)^{2k-1}},$$

where $|G|$ denotes the order of the group and $\mathrm{Irr(G)}$ the set of all irreducible characters of G.

For any finite group G define the *Witten zeta function*, $\forall t > 0$

$$\zeta^G(t) = \sum_{\chi \in \mathrm{Irr(G)}} \chi(1)^{-t}.$$

Theorem (Liebeck-Shalev, 2005)

For any (quasi-)simple algebraic group G defined over $\mathbb{F}_q$,

$$\zeta^{G(\mathbb{F}_{q^m})}(t) < C$$

uniformly for any integer $m \geq 1$ and real $t > 1$.

(Aizenbud-Avni)
By Frobenius' formula and Liebeck-Shalev's theorem on zeta function ($k \geq 2$) ($\phi_{G,k}$ denotes ϕ_k on G)

$$
\begin{aligned}
&|\phi^{-1}_{G(_{q^m}),k}(g)| \\
=&|G(\mathbb{F}_{q^m})|^{2k-1}\left|\sum_{\chi\in \mathrm{Irr(G(\mathbb{F}_{q^m}))}} \frac{\chi(g)}{\chi(1)^{2k-1}}\right| \\
\leq&|G(\mathbb{F}_{q^m})|^{2k-1}\sum_{\chi\in \mathrm{Irr(G(\mathbb{F}_{q^m}))}} \frac{1}{\chi(1)^{2k-2}} \\
=&|G(\mathbb{F}_{q^m})|^{2k-1}\zeta^{G(\mathbb{F}_{q^m})}(2k-2) \leq C|G(\mathbb{F}_{q^m})|^{2k-1}.
\end{aligned}
$$

Liebeck-Shalev theorem is not applicable to $\phi_1 = [,]$ since

$$\zeta^{G(\mathbb{F}_{q^m})}(2k-2) = \zeta^{G(\mathbb{F}_{q^m})}(0)$$

is not uniformly bounded. Instead we use a character ratio estimate

Theorem (Larsen-L, 2018)

For $g \in SL_n(\mathbb{F}_q)$ non-central,

$$\sum_{\chi \in \mathrm{Irr}(\mathrm{GL_n}(\mathbb{F}_q))} \frac{\chi(g)}{\chi(1)} = O(q).$$

(Larsen-L)
By Frobenius' formula, for $g \in SL_n(\mathbb{F}_q)$ non-central

$$\begin{aligned} &\left|\phi_1^{-1}(g)\right| \\ =&|GL_n(\mathbb{F}_q)| \sum_{\chi \in \mathrm{Irr}(GL_n(\mathbb{F}_q))} \frac{\chi(g)}{\chi(1)} \\ =&|GL_n(\mathbb{F}_q)|O(q) \\ =&O(q^{n^2+1}). \end{aligned}$$

For $\phi_1 = [,]$ on $SL_n(\mathbb{F}_q)$, this is saying

$$|\phi_1^{-1}(g)| = O(|SL_n(\mathbb{F}_q)|).$$

The above is an inequality about dimension due to

Theorem (Lang-Weil bound)

For V any variety over $\bar{\mathbb{F}}_p$ with $\dim V = f$ and e components of top dimension f,

$$|V(q)| = (e + o(1))q^f$$

for all large enough $q = p^r$.

For any $k \geq 1$ and (quasi-)simple G,

$$\dim \phi_{G,k}^{-1}(g) \leq (2k-1)\dim G.$$

Actually $\dim \phi_{G,k}^{-1}(g) = (2k-1)\dim G$ since the other direction is trivial by definition.

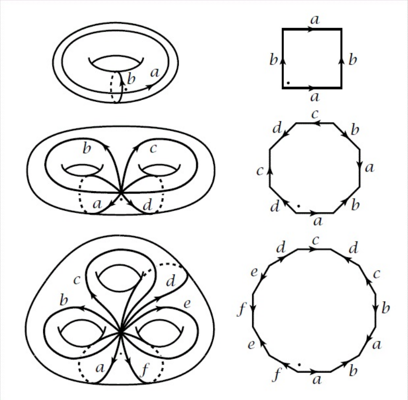
The dimension equality over $\bar{\mathbb{F}}_p$ may be turned into over $\mathbb{C}$ by Lefschetz principle in a cheap model theoretic way. To be more careful we use scheme language

Theorem (EGA IV, 9.2.6.1)

If $f : X \to S$ is a scheme morphism of finite presentation, then the function $s \mapsto dim(f^{-1}(s))$ is locally constructible.

For $g \in G(\bar{\mathbb{Q}})$, $X = \phi_{G,k}^{-1}(g)$ is definable over $\mathbb{Z}$, hence has structure morphism to $S = \operatorname{Spec} \mathbb{Z}$ and $f^{-1}((p)) = X/\mathbb{F}_p$ has some fiber of q-points as closed points.
At last, use Cohen-Macaulay machinery (*miracle flatness*) to turn the dimension equality into flatness.

Let S^g be a compact Riemann surface of genus g, which can be uniformized by identifying edges of a $4g$-gon as follows ($g \leq 3$)



- The fundamental group of S^g has the presentation

$$\pi_1(S^g) = \langle a_1, b_1, \cdots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle.$$

- For any group G, every $\rho \in \mathrm{Hom}(\pi_1(\mathrm{S}^{\mathrm{g}}), \mathrm{G})$ is determined only by the relation

$$[\rho(a_1), \rho(b_1)] \cdots [\rho(a_g), \rho(b_g)] = 1.$$

Hence

$$\mathrm{Hom}(\pi_1(\mathrm{S}^{\mathrm{g}}), \mathrm{G}) \leftrightarrow \phi_{\mathrm{g}}^{-1}(1),$$

i.e. fiber of the commutator map over the identity can be identified with the representation space of a surface group into G. What about any other fiber $\phi_g^{-1}(\mathfrak{g})$ over arbitrary $\mathfrak{g} \in G$?

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The fundamental group of $S^g \smallsetminus \{*\}$ can be presented as

$\pi_1(S^g \smallsetminus \{*\}) = \langle a_1, b_1, \cdots, a_g, b_g, c \mid [a_1, b_1] \cdots [a_g, b_g]c = 1 \rangle$.

And any $\rho \in \mathrm{Hom}(\pi_1(\mathrm{S^g} \smallsetminus \{*\}), \mathrm{G})$ is determined by

$$[\rho(a_1), \rho(b_1)] \cdots [\rho(a_g), \rho(b_g)] = \rho(c)^{-1}.$$

Hence

$$\mathrm{Hom}(\pi_1(\mathrm{S^g} \smallsetminus \{*\}), \mathrm{G}) \leftrightarrow \phi_{\mathrm{g}}^{-1}(\mathrm{G}),$$

with $\phi_g^{-1}(\mathfrak{g})$ corresponding to the representation space of $\pi_1(S^g \smallsetminus \{*\})$ into G with the free quantifier c mapped to $\mathfrak{g}$.

For any $\rho \in \mathrm{Hom}(\pi_1(\mathrm{S^g} \setminus \{*\}), \mathrm{GL_n}(\mathbb{C})), \mathrm{n} \geq 1$, we can define a complex vector bundle over $S^g \setminus \{*\}$ as follows : let X be the universal cover of $S^g \setminus \{*\}$ with deck transformation group $\pi = \pi_1(S^g \setminus \{*\})$, then $X \times \mathbb{C}^n$ is equipped with a π-action

$$\alpha \cdot (x, v) = (\alpha \cdot x, \rho(\alpha)v), \forall \alpha \in \pi, x \in X, v \in \mathbb{C}^n.$$

Since π acts properly and freely on X hence also $X \times \mathbb{C}^n$,

$$X \times \mathbb{C}^n/\pi \to X/\pi = S^g \setminus \{*\},$$

is a rank n bundle, called the *flat bundle with holonomy* ρ, denoted by E_ρ. Note that E_ρ inherits a *flat connection* (zero curvature) from the trivial bundle.

Especially when $g = 1$, $S^g \setminus \{*\}$ deformation retracts to a wedge of two circles, which has a universal cover as the Cayley graph of $F_2 = *$ as follows ($F_2 = \langle a, b \rangle$)

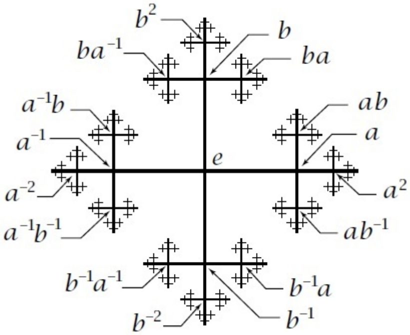

FIGURE – Hatcher page 77

Hence any flat bundle with holonomy ρ can be seen as the quotient of $T \times \mathbb{C}^n$, T the fractal tree in the picture above. In particular, F_2 is linearly realizable, say by the Sanov representation

$$\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle.$$

Hence the flat structure of E_ρ's over $S^g \setminus \{*\}$ $(g = 1)$ naturally comes from the trivial bundle $T \times \mathbb{C}^n$ quotient by linear representation of F_2. Similar for any $g \geq 2$.

Inversely, we can construct a linear representation of π from a flat vector bundle using holonomy. Let E be a rank n vector bundle over $S^g \setminus \{*\}$ and ∇ the connection on E. Given any smooth loop $\gamma : [0,1] \to S^g \setminus \{*\}$, based at x, the connection defines a linear invertible *parallel transport* $P_\gamma : E_x \to E_x$ along the loop, hence a linear transformation in $GL(E_x) = GL_n(\mathbb{C})$.

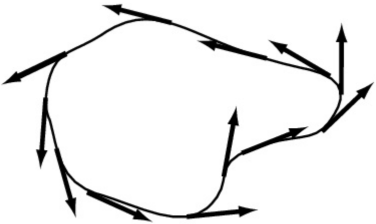

FIGURE – C. Ravelli, *Quantum Gravity*, page 11

Define

$$\mathrm{Hol_x}(\nabla) = \{\mathrm{P}_\gamma \in \mathrm{GL}(\mathrm{E_x}) \mid \gamma \text{ a loop based at x}\}.$$

Clearly any other base point gives a conjugation of the above group, hence up to isomorphism, we denote it by $\mathrm{Hol}(\nabla)$ and call it the *holonomy group* of the connection. If ∇ is flat, contractible loops gives trivial transport, resulting a surjective group homomorphism $\pi \to \mathrm{Hol}(\nabla) \subset \mathrm{GL_n}(\mathbb{C})$ which sends $[\gamma]$ to P_γ. This gives a representation of π into $GL_n(\mathbb{C})$ with image $\mathrm{Hol}(\nabla)$.

By construction, the flat bundle of holonomy E_ρ, any $\rho \in \mathrm{Hom}(\pi, \mathrm{GL_n}(\mathbb{C}))$ with the natural flat connection gives back the representation ρ through holonomy.

Geometric setting of commutator map

Thus far, we get the following identifications

$$\phi_g^{-1}(SL_n(\mathbb{C})) \leftrightarrow \mathrm{Hom}(\pi, \mathrm{GL_n}(\mathbb{C}))$$

$$\leftrightarrow \{\text{flat complex vector bundles over } S^g \setminus \{*\}\},$$

via

$$\phi_g^{-1}(\mathfrak{g}) \leftrightarrow \{\rho : \pi \to GL_n(\mathbb{C}), \rho(c) = \mathfrak{g}^{-1}\}$$

$$\leftrightarrow \{\text{complex vector bundle with } \nabla_c = \mathfrak{g}^{-1}\}.$$

THANK YOU