On geometrisation, integrability and knots

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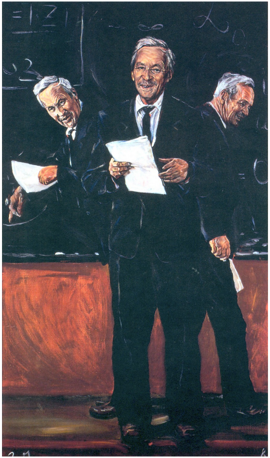
Varchenko-70 + Tarasov-60

Teachers

A.A. Shershevskiy, A.N. Kolmogorov (by D. Gordeev) and V.I. Arnold

Plan

- Geometrisation programmes in dimension 2 and 3
- Liouville-Arnold integrability revisited
- Chaos and integrability in $SL(2,\mathbb{R})$-geometry
- Geodesics on the modular 3-fold and knot theory

References

V.I. Arnold *Some remarks on flow of line elements and frames*. Sov. Math. Dokl. **138:2** (1961), 255-257.
É. Ghys *Knots and Dynamics.* Intern. Congress of Math. Vol. 1. Eur. Math. Soc., Zurich 2007, 247-277.
A. Bolsinov, A.Veselov, Y. Ye *Chaos and integrability in* $SL(2,\mathbb{R})$*-geometry.* arXiv:1906.07958.

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Given a manifold, what is the "best" metric one can introduce and in how many ways?

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In particular, for any Riemannian genus $g > 1$ surface is **conformally equivalent to a quotient** $\mathbb{H}^2/G$ **of the hyperbolic plane** $\mathbb{H}^2$ by a discrete subgroup $G \subset PSL(2,\mathbb{R})$, which is the group of motion of $\mathbb{H}^2$.

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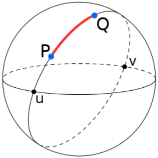
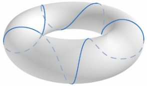
Explicit example: sphere with $n = 3$ punctures, $G = \Gamma_2 \subset PSL(2,\mathbb{Z})$ (corollary: **Picard's theorem**).

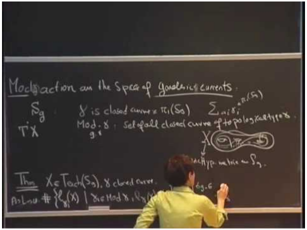
Geodesic flows on surfaces

The behaviour of geodesics on these three types of surfaces are very different.

On the round sphere all geodesics are large circles, on the flat torus they are straight winding lines, while on hyperbolic surfaces their behaviour is known to be very chaotic (**Hedlund 1930s, Anosov 1960s**).







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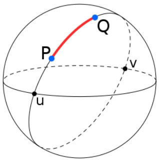

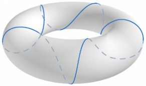

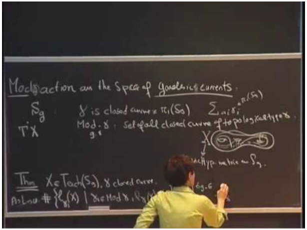

In particular, for the *modular surface* $\mathbb{H}^2/PSL(2,\mathbb{Z})$ the geodesics can be described symbolically using continued fractions (**E. Artin, 1924**).

Analogue of constant curvature in 3D are *locally homogeneous metrics*, such that any two points of M^3 have isometric neighbourhoods.

The celebrated **Thurston's geometrization conjecture** of 1982 (proved by **Perelman** in 2003) states that

Every closed 3-manifold can be decomposed into pieces such that each admits one of the following eight types of geometric structures of finite volume

$$\mathbb{E}^3,\ \mathbb{S}^3,\ \mathbb{S}^2\times\mathbb{R},\ \mathbb{H}^2\times\mathbb{R},\ Nil,\ Sol,\ \widetilde{SL(2,\mathbb{R})},\ \mathbb{H}^3,$$

where

$$Nil = \left\{\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}\right\} \quad Sol = \left\{\begin{pmatrix} e^x & 0 & y \\ 0 & e^{-x} & z \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

and $\widetilde{SL(2,\mathbb{R})}$ is the unversal cover of $SL(2,\mathbb{R})$.

Dimension 3: Thurston's geometrization programme

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What's about integrability of the corresponding geodesic flows?

Liouville-Arnold integrability

Arnold 1963: Hamiltonian system on symplectic manifold M^{2n} is integrable in Liouville sense if it has n independent integrals $F_1, \dots, F_n$ in involution.

When the joint integral level

$$M_c = \{x \in M^{2n} : F_i(x) = c_i,\ i = 1, \dots, n\}$$

is non-critical and compact, then it must be a torus T^n with quasi-periodic dynamics and in its vicinity one can introduce "action-angle" variables I_i, φ_i with $H = H(I)$.

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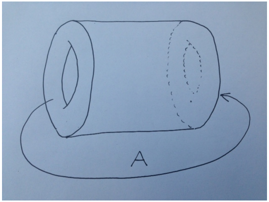
Tomei (1984), Gaifullin (2006): Toda manifold of isospectral tri-diagonal matrices is aspherical and can be used to realise cycles in Steenrod's problem!

Sol-case: chaotic critical level

In *Sol*-case the principal examples are mapping tori M_A^3 of the hyperbolic maps $A : T^2 \to T^2$, $A \in SL(2, \mathbb{Z})$ (first considered by Poincaré in 1892!):



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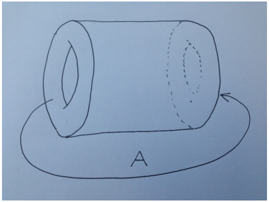

Figure: Torus mapping of A

Bolsinov and Taimanov (2000): On *Sol*-manifolds the geodesic flow is Liouville integrable in smooth category, but not in analytic.

At the degenerate level the system is chaotic (Anosov map), so the system has positive topological entropy!

$SL(2,\mathbb{R})$-case

In $SL(2,\mathbb{R})$-case the principal examples are the quotients $\mathcal{M}^3_\Gamma = \Gamma\backslash PSL(2,\mathbb{R})$, where $\Gamma \subset PSL(2,\mathbb{R})$ is a cofinite Fuchsian group, or, equivalently, the unit tangent bundles $\mathcal{M}^3_\Gamma = S\mathcal{M}^2_\Gamma$ of the hyperbolic $\mathcal{M}^2_\Gamma = \Gamma\backslash\mathbb{H}^2$.

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Bolsinov, Veselov and Ye (2019): The corresponding phase space $T^*\mathcal{M}^3_\Gamma$ contains two open regions with integrable and chaotic behaviour.

In the integrable region we have Liouville integrability with analytic integrals, while in the chaotic region the system is not Liouville integrable even in smooth category and has positive topological entropy.

I am going to explain this now in more detail.

Choose the class of (*naturally reductive*) metrics on $SL(2,\mathbb{R})$, which are **left $SL(2,\mathbb{R})$-invariant and right $SO(2)$-invariant**:

$$\langle X, Y\rangle = \alpha(\mathrm{sym}\, X, \mathrm{sym}\, Y) + \beta(\mathrm{skew}\, X, \mathrm{skew}\, Y), \;\; \alpha > 0 > \beta,$$

$$(X, Y) := \mathrm{Tr}\, XY, \;\; \mathrm{skew}\, X := (X - X^\top)/2 \in so(n), \;\; \mathrm{sym}\, X := (X + X^\top)/2.$$

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Setting $\alpha = 2$, we have the inner product with

$$|\Omega|^2 = 4(u^2 + vw) + k(v - w)^2, \quad k = 1 - \frac{\beta}{\alpha} > 1$$

on the Lie algebra

$$\Omega = \begin{pmatrix} u & v \\ w & -u \end{pmatrix} \in sl(2,\mathbb{R}).$$

$SL(2,\mathbb{R})$ as unit tangent bundle of hyperbolic plane

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Explicitly we have the identification

$$g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R}) \longrightarrow (z = \frac{ai+b}{ci+d}, \xi = \frac{i}{(ci+d)^2}) \in S\mathbb{H}^2,$$

where $\mathbb{H}^2$ is realised as the upper half-plane $z = x + iy$, $y > 0$ with the hyperbolic metric $ds^2 = dzd\bar{z}/y^2$.

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In coordinates $x, y, \varphi = \arg \xi$ the metric has the form

$$ds^2 = \frac{dx^2 + dy^2}{y^2} + (k-1)(d\varphi + \frac{dx}{y})^2,$$

which is the generalised **Sasaki metric** on $S\mathbb{H}^2$, considered by **Nagy (1977)**. Sasaki metric corresponds to $k = 2$ and can be considered as the "best" one.

Euler-Poincare equations and formulae for geodesics on $SL(2,\mathbb{R})$

The general Euler-Poincare equations of the corresponding geodesic flow have

$$\dot{M} = [M, \Omega],$$

where $\Omega := g^{-1}\dot{g} \in \mathfrak{g}$ and $M \in \mathfrak{g}^* \cong \mathfrak{g}$ is determined by $(\Omega, M) = \langle \Omega, \Omega \rangle$.

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In our case we have $2M = (\alpha+\beta)\Omega + (\alpha-\beta)\Omega^\top$, so the Euler-Poincare equations have the form

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The geodesics on $SL(2,\mathbb{R})$ with $\Omega(0) = \Omega_0$ can be explicitly given by

$$g(t) = g(0)e^{tX_0}e^{tY_0},$$

where

$$X = \frac{1}{\alpha}M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad Y = \frac{\alpha - \beta}{2\beta}\begin{pmatrix} 0 & b - c \\ c - b & 0 \end{pmatrix}.$$

Projection of geodesics on hyperbolic plane

We have natural projection of $SL(2,\mathbb{R}) = S\mathbb{H}^2$ to $\mathbb{H}^2$. Introduce

$$\Delta = -\det X = a^2 + bc.$$

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Theorem

The projection of the geodesics on $SL(2,\mathbb{R})$ to $\mathbb{H}^2$ are curves with constant geodesic curvature

$$\kappa = \frac{b-c}{\sqrt{4a^2+(b+c)^2}},$$

which are circles if $\kappa^2 > 1$ (or equivalently, if $\Delta > 0$), or arcs of circles when $\kappa^2 \leq 1$ (or $\Delta \leq 0$) and can be interpreted as magnetic geodesics on $\mathbb{H}^2$.

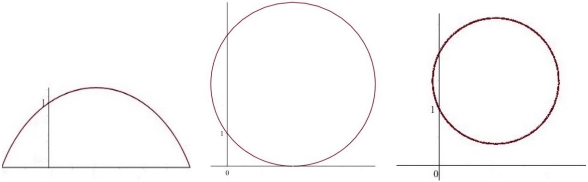

Figure: Projection of geodesics with $\Delta > 0$, $\Delta = 0$, $\Delta < 0$ respectively)

We have two obvious left-invariant Poisson commuting integrals of geodesic flow on $G = SL(2, \mathbb{R})$: Hamiltonian

$$H = \frac{1}{2}(\Omega, M) = \frac{\alpha}{4\beta}(\beta[4a^2 + (b+c)^2] - \alpha(b-c)^2)$$

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Note that any other such function generates the left shifts and gives an additional integral of the system. Thus the invariant tori of the system have dimension 2 in agreement with the previous picture.

Liouville integrability for Fuchsian quotients M^3_Γ

Let $\Gamma \subset PSL(2,\mathbb{R})$ be a Fuchsian group such that $\Gamma\backslash\mathbb{H}^2 = \mathcal{M}^2_\Gamma$ has finite area and consider the quotient $\mathcal{M}^3_\Gamma = \Gamma\backslash PSL(2,\mathbb{R}) = S\mathcal{M}^2_\Gamma$.

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It is known that this action is discrete if $\Delta = \delta < 0$ (which is a model of $\mathbb{H}^2$) and has some dense orbits if $\Delta = \delta > 0$ (**Hedlund, Dal'Bo**).

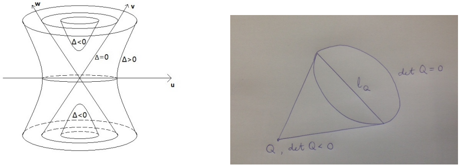

Figure: $sl(2,\mathbb{R})$-symplectic leaves and Klein's correspondence

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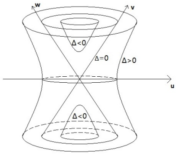

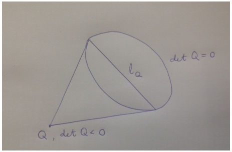

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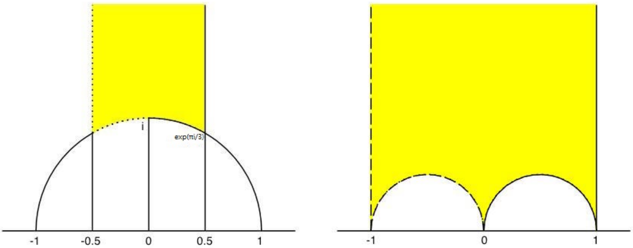
Corollary: The geodesic flow on $T^*\mathcal{M}^3_\Gamma$ has no smooth right-invariant integrals F independent from Δ in the part of the phase space $T^*\mathcal{M}^3_\Gamma$ with $\Delta \geq 0$. In the domain $\Delta < 0$ we can use any real analytic automorphic function as the additional third analytic integral F.

Special case: modular groups

Consider now the special case of modular group $\Gamma = PSL(2, \mathbb{Z})$ and its principal congruence subgroup Γ_2.



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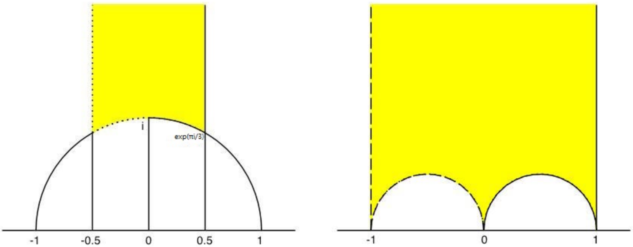

Figure: The fundamental domains of Γ and Γ_2

In the first case the quotient $\mathcal{M}^2 = PSL(2, \mathbb{Z})\backslash\mathbb{H}^2$ is the orbifold with two orbifold points corresponding to the elliptic elements in $PSL(2, \mathbb{Z})$ of order 2 and 3 respectively. In the second case we have the 3-point punctured sphere.

Modular 3-fold and knots

Let $\Gamma = PSL(2,\mathbb{Z})$ be the modular group and consider the *modular 3-fold*

$$\mathcal{M}^3 = SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R}).$$

In that case in the domain $\Delta < 0$ we can write down the third additional analytic integral explicitly in terms of the J-function.

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There is a remarkable observation due to **Quillen (1970s):**

$$\mathcal{M}^3 = SL(2, \mathbb{Z})/SL(2, \mathbb{R}) = S^3 \backslash \mathcal{K},$$

where $\mathcal{K}$ is the trefoil knot:

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Milnor, 1972: Note first that $\mathcal{M}^3 = SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ can be interpreted as the moduli space of the elliptic curves $\mathbb{C}/\mathcal{L}$ up to real scaling. The corresponding $\wp$-function satisfies the Weierstrass equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

which defines an elliptic curve if and only if the discriminant

$$D = g_2^3 - 27g_3^2 \neq 0.$$

The intersection of the unit sphere $S^3 \subset \mathbb{C}^2(g_2, g_3)$ with the set $D = 0$ is $(2,3)$-torus ($=$ trefoil) knot.

Quillen's proof

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Alternatively, the projection $\mathcal{M}^3 \to \mathcal{M}^2 = \mathbb{H}^2/PSL(2,\mathbb{Z})$ is the Seifert fibration with two singular fibres corresponding to orbifold points of order 2 and 3 of $\mathcal{M}^2$. The missing Hopf fibre over infinity is thus (2,3)-torus knot.

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Modular knots and Lorenz system

E. Artin, 1924: Periodic geodesics on modular surface $\mathcal{M}^2$ are labelled by integer binary quadratic forms Q. Their lifts to $\mathcal{M}^3 = S\mathcal{M}^2$ form certain knots called by Ghys *modular*.

Birman and Williams (1983), Ghys (2006): Modular knots are exactly those, which appear as periodic orbits in the celebrated Lorenz system

$$\begin{cases} \dot{x} = \sigma(-x+y) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases}, \quad \sigma = 10,\ b = 8/3,\ r = 28.$$

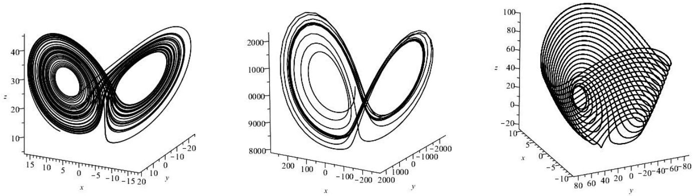

Figure: The Lorenz trajectories for $r = 28$, 10000 and $r = \infty$

Integrable limit and cable knots

Consider the integral

$$\mathcal{C} = \frac{(b-c)^2}{4a^2 + (b+c)^2} = \frac{\beta H - \alpha\beta\Delta}{\beta H - \alpha^2\Delta}$$

of the geodesic flow on $\mathcal{M}^3$. We have seen that the system is integrable if $\mathcal{C} > 1$ and non-integrable otherwise.
When $\mathcal{C} = 0$ we have the lifts of the geodesics on the modular surface $\mathcal{M}^2$ considered by Ghys. It is natural to ask what happens when $\mathcal{C} > 1$.

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BVY(2019): The periodic geodesics on modular 3-fold $\mathcal{M}^3_\Gamma$ with sufficiently large values of $\mathcal{C}$ represent the trefoil cable knots in $S^3 \setminus \mathcal{K}$.
Any cable knot of trefoil can be realised in such a way.

Geometric classification of knots

Thurston (1978): Every knot is either torus, or a satellite, or hyperbolic knot.

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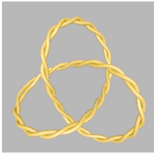
Torus knot $K_{p,q}$ specified by a pair of coprime integers p and q lies on the surface of a solid torus in $\mathbb{R}^3$, winding p times around the axis of rotation of the torus and q times around the central circle of the torus. Trefoil knot $\mathcal{K} = K_{2,3}$.

Satellite knots can be get in the following way: let K_1 be a knot inside an unknotted solid torus and knot the torus in the shape of another knot K_2. In the special case of K_1 being a torus knot, we have *cable knots of K_2*.



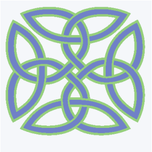


Figure: Trefoil knot $\mathcal{K}$, its $(2, 33)$ cable knot and celtic satellite knot

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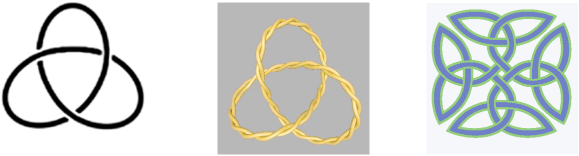


Figure: Trefoil knot $\mathcal{K}$, its $(2, 33)$ cable knot and celtic satellite knot

Knot is *hyperbolic* if its complement admits hyperbolic structure. Examples: modular knots.

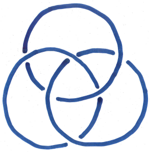
Complements to the torus knots admit $SL(2, \mathbb{R})$-structure, while for the satellite knots they do not admit any of the 8 geometric structures.

Congruence subgroup Γ_2

Let $\Gamma_2 \subset SL(2, \mathbb{Z})$ consist of matrices congruent to the identity modulo 2:

$$\mathcal{M}_2^3 = \Gamma_2 \backslash SL(2, \mathbb{R}) \cong S^3 \backslash \mathcal{L},$$

where $\mathcal{L}$ is the **Hopf 3-link**



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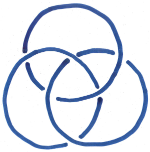

In the integrable domain $\Delta < 0$ we can write the third analytic integral explicitly in terms of the λ-function. When the ratio frequencies of the motion is rational

$$\frac{\omega_1}{\omega_2} = \frac{\beta - \alpha}{2\beta} \frac{|b - c|}{\sqrt{-\Delta}} = \frac{p}{q}, \quad p, q \in \mathbb{Z},$$

for large $\mathcal{C}$ we have the invariant torus filled by the torus knots $K_{p,q}$.
Thus in the integrable limit the hyperbolic modular knots are replaced by the torus knots (cf. Lorenz system).

Some open questions

- Study the types of knots in $\mathcal{M}^3 = S^3 \setminus \mathcal{K}$ outside the integrable limit. *Can any knot be realised this way?*

  Volume Conjecture **(Kashaev 1997, Murakami et al 2002)**: for hyperbolic knots

  $$Vol(S^3 \backslash K) = 2\pi \lim_{N \to \infty} \frac{\ln |J_N(K)|}{N},$$

  where $J_N(K)$ is the *Jones polynomial* of K evaluated at $e^{2\pi i/N}$.
  Volumes for modular knots: **Brandts, Pinsky, Silberman 2017**

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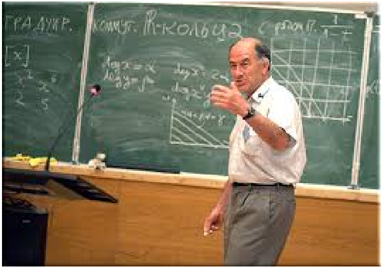
  For the modular surface $\mathcal{M}^2 = \mathbb{H}^2/SL(2,\mathbb{Z})$: **Maas, Selberg, Faddeev, Hejhal (1940-70s)**.

Sasha and Vitaly: Many Happy Returns!



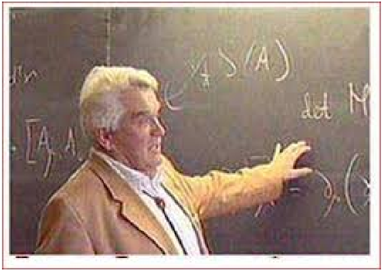


Figure: Arnold and Faddeev in action