Vertex operators presentation of generalized Hall-Littlewood polynomials

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Hall-Littlewood polynomial

Let $\lambda \vdash n$. Hall-Littlewood polynomial:

$$P_\lambda(x_1, \dots, x_n; t) = \left(\prod_{i \geq 0} \prod_{j=1}^{m(i)} \frac{1-t}{1-t^j} \right) \sum_{\sigma \in S_n} \sigma \left(x_1^{\lambda_1} \dots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right)$$

When $t = 1$,

$$P_\lambda(x_1, \dots, x_n; 1) = s_\lambda(x_1, \dots, x_n)$$

– Schur symmetric functions.

Generalizations

Schur symmetric functions
$s_\lambda(x)$

$\downarrow$

Hall-Littlewood polynomials
$P_\lambda(t;x)$

$\swarrow \quad \searrow$

Macdonald polynomials $P_\lambda(t,q;x)$ $\qquad$ $P_\lambda(a_1,a_2,\ldots;b_1,b_2,\ldots;x)$

$$g_n = \prod_{i=1}^{\infty} \frac{1-tq^i}{1-q^i} P_{(n)}(q,t;x) \qquad = \qquad P_{(n)}(q,q^2,\ldots;tq,tq^2,\ldots;x)$$

Topics to discuss

–Vertex operators presentations of symmetric functions s_λ imply properties of $P_\lambda(a_1, a_2, \dots; b_1, b_2, \dots)$, of Hall-Littlewood polynomials $P_\lambda(t; x)$ in particular.

– simple proofs of generalized Cauchy identities decompositions.

– action of charged free fermions on

$$\Lambda \otimes \mathbb{C}[a_1, a_2, \dots b_1, b_2, \dots] \otimes \mathbb{C}[z, z^{-1}]$$

and examples of solutions of KP hierarchy in this ring.

Notations

The ring of symmetric functions $\Lambda = \Lambda[x_1, x_2, \dots]$
Complete symmetric functions $h_r = s_{(r)}$

$$h_r(x_1, x_2 \dots) = \sum_{1 \le i_1 \le \cdots \le i_r < \infty} x_{i_1} \dots x_{i_r}.$$

Elementary symmetric functions $e_r = s_{(1^r)}$

$$e_r(x_1, x_2 \dots) = \sum_{1 < i_1 < \cdots < i_r < \infty} x_{i_1} \dots x_{i_r}.$$

Power sums: $p_k(x_1, x_2, \dots) = \sum_i x_i^k$.

Generators and scalar product in Λ

$$\Lambda = \mathbb{C}[h_1, h_2, \dots] = \mathbb{C}[e_1, e_2, \dots] = \mathbb{C}[p_1, p_2, \dots].$$

Λ has a natural scalar product, where

$$< s_\lambda, s_\mu >= \delta_{\lambda,\mu}.$$

Then for the operator of multiplication by $f \in \Lambda$ one can define the adjoint operator $f^\perp$ by the standard rule:

$$< f^\perp g, w >=< g, fw >$$

with $f, g, w \in \Lambda$.

Generating functions

Define formal distributions of operators acting on Λ

$$H(u) = \sum_{k\geq 0} \frac{h_k}{u^k}, \qquad E(u) = \sum_{k\geq 0} \frac{e_k}{u^k}.$$

$$E^{\perp}(u) = \sum_{k\geq 0} e_k^{\perp} u^k, \quad H^{\perp}(u) = \sum_{k\geq 0} h_k^{\perp} u^k.$$

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Note that

$$H(u) = exp\left(\sum_{n\geq 1} \frac{p_n}{n}\frac{1}{u^n}\right), \quad E(u) = exp\left(-\sum_{n\geq 1} \frac{(-1)^n p_n}{n}\frac{1}{u^n}\right).$$

$$E^{\perp}(u) = exp\left(-\sum_{k\geq 1}(-1)^k \frac{\partial}{\partial p_k} u^k\right), \quad H^{\perp}(u) = exp\left(\sum_{k\geq 1} \frac{\partial}{\partial p_k} u^k\right).$$

Commutation relations

Note $H(u)E(-u) = 1$.

$$\left(1-\frac{u}{v}\right) E^{\perp}(u)E(v) = E(v)E^{\perp}(u),$$

$$\left(1-\frac{u}{v}\right) H^{\perp}(u)H(v) = H(v)H^{\perp}(u),$$

$$H^{\perp}(u)E(v) = \left(1+\frac{u}{v}\right) E(v)H^{\perp}(u),$$

$$E^{\perp}(u)H(v) = \left(1+\frac{u}{v}\right) H(v)E^{\perp}(u).$$

– as series expansions in powers of $u^k v^{-m}$ for $k, m \geq 0$.

Charged free fermions acting on symmetric functions

One defines boson Fock space $\mathcal{B} = \oplus_{m\in\mathbb{Z}}\mathcal{B}^{(m)}$,

$$\mathcal{B}^{(m)} = z^m \cdot \mathbb{C}[p_1, p_2, \dots] = z^m \cdot \Lambda$$

Define

$$\Phi^{\pm}(u) = \sum_k \Phi_k u^{\mp k}$$

by

$$\Phi^{+}(u)|_{\mathcal{B}^{(m)}} = zu^{-m-1}H(u)E^{\perp}(-u),$$
$$\Phi^{-}(u)|_{\mathcal{B}^{(m)}} = z^{-1}u^{m-1}E(-u)H^{\perp}(u).$$

Then vertex operators $\Phi^{\pm}(u)$ provide **the action of algebra of charged free fermions** on the boson Fock space.

$$\Phi_k^{\pm}\Phi_l^{\pm} + \Phi_l^{\pm}\Phi_k^{\pm} = 0,$$
$$\Phi_k^{+}\Phi_l^{-} + \Phi_l^{-}\Phi_k^{+} = \delta_{k,l}.$$

$$\Phi^{\pm}(u)\Phi^{\pm}(v) + \Phi^{\pm}(v)\Phi^{\pm}(u) = 0,$$
$$\Phi^{+}(u)\Phi^{-}(v) + \Phi^{-}(v)\Phi^{+}(u) = \delta(u,v).$$

Here $\delta(u,v) = \sum_{k\in\mathbb{Z}} \frac{u^k}{v^{k+1}}$ is formal delta distribution.

Vertex operator presentation of Schur functions

One has:

$$\Phi^{+}(u_1)\ldots\Phi^{+}(u_l)\,(1) = z^l u_1^{-l} \ldots u_l^{-1} Q(u_1, .., u_l),$$

where

$$Q(u_1,\ldots,u_l) = \prod_{1\le i<j\le l}\left(1-\frac{u_i}{u_j}\right)\prod_{i=1}^{l} H(u_i).$$

[N.Jing 1991] Consider the series expansion of the rational function

$$Q(u_1,\ldots,u_l) = \sum_{(\lambda_1,\ldots,\lambda_l)\in\mathbb{Z}^l} Q_\lambda\, u_1^{-\lambda_1}\ldots u_l^{-\lambda_l}$$

in the region $|u_1| < \cdots < |u_l|$. For any partition $\lambda = (\lambda_1,\ldots,\lambda_l)$ the coefficient of $u_1^{-\lambda_1}\ldots u_l^{-\lambda_l}$ is exactly Schur symmetric function:

$$Q_\lambda = s_\lambda.$$

Generalized vertex operators

Consider two collections of parameters $(a_1, a_2, \ldots,)$ and $(b_1, b_2, \ldots,)$. Set

$$a(x) = \prod_i (1 - a_i x) \qquad b(x) = \prod_i (1 - x b_i).$$

Set

$$T(u) = \prod_i H\left(\frac{u}{a_i}\right) \prod_i E\left(\frac{-u}{b_i}\right).$$

Then

$$T^{-1}(u) = \prod_i E\left(\frac{-u}{a_i}\right) \prod_i H\left(\frac{u}{b_i}\right).$$

Define generalized vertex operators

$$\Gamma^{+}(u) = T(u)E^{\perp}(-u), \quad \Gamma^{-}(u) = T^{-1}(u)H^{\perp}(u).$$

These are formal distributions with coefficients – operators acting on $\Lambda \otimes \mathbf{Q}(a_1, a_2, \ldots b_1, b_2, \ldots)$. Then

$$\Gamma^{+}(u_1) \ldots \Gamma^{+}(u_l)(1) = \prod_{i<j} \frac{a(u_i/u_j)}{b(u_i/u_j)} \prod_{i=1}^{l} T(u_i).$$

Similarly, one can prove that

$$\Gamma^{-}(u_1) \ldots \Gamma^{-}(u_l)(1) = \prod_{i<j} \frac{a(u_i/u_j)}{b(u_i/u_j)} \prod_{i=1}^{l} T^{-1}(u_i).$$

For $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{Z}^l$ define

$$T_\alpha \in \Lambda \otimes \mathbf{Q}(a_1, a_2, \ldots ; b_1, b_2, \ldots)$$

as coefficients of the expansion

$$T(u_1, \ldots, u_l) = \prod_{i<j} \frac{a(u_i/u_j)}{b(u_i/u_j)} \prod_{i=1}^{l} T(u_i) = \sum_\alpha T_\alpha u^\alpha,$$

in the region $|u_1| < |u_2| \cdots < |u_l|$. Thus,

$$T_\alpha = \Gamma^+_{\alpha_1} \ldots \Gamma^+_{\alpha_l}(1)$$

Example. Let $a(x) = 1 - x$, $b(x) = 1 - tx$. Then

$$T(u_1, \dots, u_l) = \prod_{1 \le i < j \le l} \frac{u_j - u_i}{u_j - u_i t} \prod_{i=1}^{l} H(u_i) E(-u_i/t).$$

When $\alpha = (\alpha_1, \dots, \alpha_l)$ is a partition, the coefficient T_α of $u_1^{\alpha_1} \dots u_l^{\alpha_l}$ coincides with a **Hall-Littlewood** symmetric function [N. Jing 1992].

(**Schur functions** case: $a(x) = 1 - x$, $b(x) = 1$).

Example. Suppose all non-zero parameters $(a_1, \ldots, a_M)$ are different, and there is the same number of non-zero parameters $(b_1, \ldots, b_M)$. Then

$$T_s(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{k=1}^{M} x_i^s a_k^s (1 - b_k/a_k) \prod_{(r,j)\neq(k,i)} \frac{a_k x_i - b_r x_j}{a_k x_i - a_r x_j}.$$

$T_\alpha = T_\alpha(a_1, a_2, \dots; b_1, b_2, \dots)$ are defined through symmetric functions. We can use known properties of symmetric functions to prove properties of T_α.

Example. Stability property of T_α easily follows from stability property of symmetric functions.

Example. Familiar commutation relations of vertex operators in special cases.

Familiar commutation relations in special cases

Example Let $a(x) = 1 - x$, and $b(x)$ – any polynomial.
Then vertex operators $\Gamma^{\pm}(u)$ satisfy the commutation relations that generalize charged free fermions relations:

$$wb(u/w)\Gamma^{\pm}(u)\Gamma^{\pm}(w) + ub(w/u)\Gamma^{\pm}(w)\Gamma^{\pm}(u) = 0,$$

$$b(w/u)\Gamma^{-}(u)\Gamma^{+}(w) + b(u/w)\Gamma^{+}(w)\Gamma^{-}(u) = \prod_{i=1}^{M}(1 - b_i)^2\delta(u, w) \cdot Id.$$

Example

If $b(x)$ and $a(x)$ satisfy condition $(1-x)b(x) = (1-xt)a(x)$, then $\Gamma^{\pm}(u)$ satisfy commutation relations of twisted fermions.

$$\left(1-\frac{ut}{v}\right)\Gamma^{\pm}(u)\Gamma^{\pm}(v) + \left(1-\frac{vt}{u}\right)\Gamma^{\pm}(v)\Gamma^{\pm}(u) = 0,$$
$$\left(1-\frac{vt}{u}\right)\Gamma^{+}(u)\Gamma^{-}(v) + \left(1-\frac{ut}{v}\right)\Gamma^{-}(v)\Gamma^{+}(u) = \delta(u,v)(1-t)^2.$$

(Hall-Littlewood case is $a(x) = (1-x)$, $b(x) = (1-xt)$)

Orthogonality - Cauchy identities

Notation: $u(\bar{x}) = u(x_1, x_2 \dots)$ symmetric function in variables x_i's.

Recall that

$$\prod_{i,j} \frac{1}{1 - x_i y_i} = \sum_{\lambda} u_\lambda(\bar{x}) v_\lambda(\bar{y})$$

represents an orthogonality condition on $\{u_\lambda(\bar{x})\}$, $\{v_\lambda(\bar{x})\}$, For example

$$\prod_{i,j} \frac{1}{1 - x_i y_i} = \sum_{\lambda} s_\lambda(\bar{x}) s_\lambda(\bar{y}) \quad \Leftrightarrow \quad < s_\lambda, s_\mu >= \delta_{\lambda,\mu}.$$

Applications: constructions of probability measures on the Young graph; MacMahon's generating function of random plane partitions, solving integrable hierarchies, etc.

Generalizations are used to construct important families of symmetric functions (e.g. Macdonald polynomials).

We substitute this identity by

$$\prod_{i,j=1}^{\infty} \frac{b(x_i y_j)}{a(x_i y_j)} = \sum_{\lambda} u_\lambda(\bar{x}) v_\lambda(\bar{y}).$$

for some

$$u_\lambda(\bar{x}) = u_\lambda(a_1, \ldots, b_1, \ldots; x_1, x_2 \ldots),$$

$$v_\lambda(\bar{x}) = v_\lambda(a_1, \ldots, b_1, \ldots; x_1, x_2 \ldots).$$

Recall

$$T(u) = \prod_i H\left(\frac{u}{a_i}\right) \prod_i E\left(\frac{-u}{b_i}\right) = \sum_{k\geq 0} T_k u^k.$$

$$H(u) = \sum_{k\geq 0} h_k u^k.$$

Consider a homomorphism $\varphi : \Lambda \to \Lambda \otimes \mathbf{Q}(a_1, a_2, \dots; b_1, b_2, \dots)$ defined on the generators of Λ by

$$\varphi(1) = 1, \qquad \varphi(h_k) = T_k.$$

Lemma. The images of power sums and Schur functions under φ are

$$\varphi(p_n) = \sum_j \left(a_j^n - b_j^n\right) p_n,$$

$$\varphi(s_\lambda) = \det[T_{\lambda_i - i + j}].$$

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Proposition. Suppose $\{u_\lambda\}, \{v_\lambda\}$ - is a pair of orthogonal to each other bases of Λ with respect to canonical scalar form on the ring symmetric functions:

$$< u_\lambda, v_\mu >= \delta_{\lambda,\mu}.$$

Then

$$\prod_{i,j=1}^{\infty} \frac{b(x_i y_j)}{a(x_i y_j)} = \sum_\lambda \varphi(u_\lambda[x]) v_\lambda[y].$$

This proposition immediately provides us several decompositions:

$$\prod_{i,j=1}^{\infty} \frac{b(x_i y_j)}{a(x_i y_j)} = \sum_{\lambda} T_{\lambda_1}[x] \dots T_{\lambda_l}[x] \, m_\lambda[y],$$

where m_λ are monomial symmetric functions.

$$\prod_{i,j=1}^{\infty} \frac{b(x_i y_j)}{a(x_i y_j)} = \sum_{\lambda} z_\lambda(a,b)^{-1} p_\lambda[x_1, x_2, \dots] p_\lambda[y_1, y_2, \dots],$$

Here

$$z_\lambda(a,b) = z_\lambda \prod_{i=1}^{l} (a_j^{\lambda_i} - b_j^{\lambda_i})^{-1},$$

$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$,
m_i is the number of parts of λ equal to i.

$$\prod_{i,j=1}^{\infty} \frac{b(x_i y_j)}{a(x_i y_j)} = \sum_{\lambda} S_\lambda[x_1, x_2 \ldots] s_\lambda[y_1, y_2, \ldots],$$

where s_λ – classial Schur symmetric functions, and

$$S_\lambda = \det[\varphi(h_{\lambda_i - i + j}(\bar{x}))] = \det[T_{\lambda_i - i + j}(\bar{x})].$$

Corollary. S_λ's are solutions of the KP hierarchy.

τ-functions of the KP bilinear equation

[M. Sato, M.Jimbo, T. Miwa, E. Date, M.Kashiwara, (...)]

The KP equation:

$$\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left(u_t - \frac{3}{2}uu_x - \frac{1}{4}u_{xxx}\right).$$

The KP equation in terms of the Hirota derivatives:

$$(D_1^4 + 3D_2^2 - 4D_1D_3)\tau \cdot \tau = 0.$$

Bilinear form of the KP hierarchy: look for solutions

$$\tau = \tau(p_1, p_2, p_3, \dots)$$

of the identity

$$\Omega\,(\tau \otimes \tau) = 0,$$

where

$$\Omega = \sum_{k \in \mathbb{Z}} \Phi_k^{+} \otimes \Phi_k^{-}.$$

$\Phi^{\pm}$ – charged free fermions acting on $\mathcal{B} = \Lambda \otimes \mathbb{C}[z, z^{-1}]$.

Schur function $s_\lambda \in \Lambda = \mathbb{C}[p_1, p_2, \dots]$ - is a solution of KP hierarchy.

Example. Set

$$a(x) = \prod_{i=1}^{\infty}(1 - q^i x)$$

$$b(x) = \prod_{i=1}^{\infty}(1 - tq^i x)$$

to obtain Cauchy identity that corresponds to scalar product in the definition of Macdonald polynomials.

$$\prod_{i=1}^{\infty} \frac{(1 - tq^i x)}{(1 - q^i x)} = \sum_{\lambda} u_\lambda(\bar{x}) v_\lambda(\bar{y}).$$

Macdonald polynomials are also eigenfunctions of an operator E. This E can be wirtten as

$$\eta_0 = (t-1)E + 1,$$

where

$$\eta(z) = H(z)\prod_i E(-t^i z)\, E^{\perp}(z)\prod_i H(q^i z).$$

[B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi, S. Yanagida 2009], [S.Koshida 2019]

Corresponding generalized vertex operators in our picture are

$$\Gamma^{+}(u) = \prod_{i=0}^{\infty} H\left(\frac{u}{q^i}\right) E\left(\frac{-u}{tq^i}\right) E^{\perp}(-u)\,,$$

$$\Gamma^{-}(u) = \prod_{i=0}^{\infty} E\left(\frac{-u}{q^i}\right) H\left(\frac{u}{tq^i}\right) H^{\perp}(u)\,.$$

appear in [Foda-Wheeler 2009] on generating functions of weighted plane partitions.

Then there is an action of **charged free fermions** on

$$\Lambda \otimes \mathbf{Q}(q,t) \otimes \mathbb{C}[z^{\pm 1}].$$

$$\Phi^{+}(q,t,u) = \prod_{i\geq 0} H(q^i u)E(-tq^i u)E^{\perp}\left(-u/t^i\right) H^{\perp}\left(u/qt^i\right)$$

$$\Phi^{-}(q,t,u) = \prod_{i\geq 0} E(-q^i u)H(tq^i u)H^{\perp}\left(u/t^i\right) E^{\perp}\left(-u/qt^i\right)$$

In this case

$$T_n = \prod_{i=1}^{\infty} \frac{1 - tq^i}{1 - q^i} P_{(n)}(q, t; x),$$

where $P_{(n)}(q, t; x)$ – Macdonald polynomials with $\lambda = (n)$.

Symmetric functions $S_\lambda = \det[T_{\lambda_i - i + j}(\bar{x})]$ are solutions of the corresponding bilinear identity and the KP hierarchy.

Vitaly and Alexander, Happy Birthday!