Noncommutative geometry and special functions

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Motivation

Many classical special functions are closely related to moduli spaces of differential equations: the hypergeometric equation lives in a 0-dimensional moduli space (i.e., is uniquely determined by singularities), the Painlevé transcendents describe flows in 2-dimensional moduli spaces (with similar flows in more complicated cases).

There are discrete analogues of these equations, culminating in elliptic hypergeometric functions (satisfying a difference equation on an elliptic curve), and elliptic Painlevé. So want to understand moduli spaces of (symmetric) elliptic difference equations, as well as their degenerations.

What are we classifying?

Differential equations:

$$v' = Av, \quad A \in \mathsf{gl}_n(k(C))$$

(Ordinary) Difference equations:

$$v(z+1) = A(z)v(z), \quad A \in \mathsf{GL}_n(k(z))$$

q-Difference equations:

$$v(qz) = A(z)v(z), \quad A \in \mathsf{GL}_n(k(z))$$

Also elliptic equations and symmetric difference equations (add condition $v(-z) = v(z)$ or $v(1/z) = v(z)$ with appropriate consistency/cocycle condition).

Example: Painlevé VI

If we consider second-order Fuchsian differential equations with four singular points with specified exponents, then the resulting moduli space is a (rational) elliptic surface, with one fiber (of type I_0^*/D_4) removed. The fundamental group of the complement of the singular locus is independent of the locations of the singularities, and the moduli space is biholomorphic to the moduli space of monodromy representations. If we fix a given monodromy representation and take the singularities 0, 1, ∞, λ, then the equation can be expressed in terms of a function $u(\lambda)$ satisfying a second-order nonlinear differential equation (the Painlevé VI equation).

This both degenerates (to mildly non-Fuchsian equations) and generalizes (to difference or q-difference equations). There are also *discrete* isomonodromy deformations, in which the exponents are shifted by integers. I will focus on these, both because they survive to more general equations and because I don't know how to construct (or even interpret geometrically) the continuous deformations in full generality. . .

Why geometry?

Differential equations are (in general) connections on vector bundles, which are hard* to deal with. But when the bundle is trivial, a connection is just a matrix of differentials ($A(z)dz$). Leads to notion of "Higgs bundle": a vector bundle W with a map $\phi : W^* \to W^* \otimes \omega_C$.

If ϕ is meromorphic, we can factor it:

$$\phi \otimes \omega_C^{-1} = B_\infty^{-t} B_0^t$$

with $B_\infty : V \to W$, $B_0 : V \to W \otimes \omega_C$, which we can encode as $B : W^* \to V^* \otimes (\mathcal{O}_C \oplus \omega_C)$.

*For me, at least!

We may write $\mathcal{O}_C \oplus \omega_C \cong \rho_* \mathcal{O}_X(1)$ where X is the ruled surface $\mathbb{P}(\mathcal{O}_C \oplus \omega_C)$, so by adjunction and twisting:

$$B : \rho^* V \otimes \mathcal{O}_X(-1) \to \rho^* W$$

This is a relative minimal resolution of its cokernel, so it's equivalent to consider $M = \text{coker}(B)$.

So Higgs bundles are classified by sheaves on $\mathbb{P}(\mathcal{O}_C \oplus \omega_C)$ which are transverse to the section "at infinity"*; the singularities are determined by how M meets the (anticanonical) double section.

*Why "at infinity"? Removing it gives the (symplectic) total space of the cotangent bundle.

More generally, given (nearly*) any Poisson ruled surface, there's a similar interpretation of sheaves as as not-quite differential or difference equations.† Conversely, every case above apart from differential equations on higher-genus curves and nonsymmetric elliptic difference equations corresponds in this way to sheaves on rational surfaces.

Symmetric equations arise when the anticanonical curve C_α is integral, in which case B_∞ and B_0 are related by an involution on the curve. If the curve is reduced but reducible, we get a difference equation on one of the two components.

*There are some weird cases in characteristic 2

†See arXiv:1307.4033, "Generalized Hitchin systems on rational surfaces"

The singularities of the equation correspond to places where the sheaf meets the kernel of the Poisson structure (the "anticanonical curve"); if we resolve these by blowing up, we get a compactly supported sheaf on the symplectic part of the surface. Gives a symplectic moduli space (pace Tyurin, Bottacin, Hurtubise/Markman), and a family of commuting symplectomorphisms (twist by a line bundle).

Main problems: (1) the moduli space is wrong (they aren't actually connections), (2) the corresponding (Hitchin-type) integrable system is autonomous/isospectral.

This does give useful results, though: if the moduli space of Higgs bundles is rational, this will be inherited by the corresponding component of the moduli space of equations (and if explicit leads to an explicit Lax pair)

Why *noncommutative* geometry?

Differential equations $\sim$ $\mathcal{D}$-modules, so can hope for difference equations to be modules (or sheaves) on something nice. Since $\mathcal{D}$-modules on C $\sim$ sheaves on a noncommutative version of T^*C, and work with Okounkov constructed elliptic Painlevé as an action on a moduli space of sheaves on a noncommutative $\mathbb{P}^2$, maybe difference equations also correspond to sheaves on a noncommutative rational surface?

Noncommutative projective surfaces

Note that a noncommutative deformation of a commutative surface determines a Poisson structure, or (essentialy equivalently) an anticanonical curve. On a rational surface, this is generically smooth genus 1 $\Rightarrow$ symmetric elliptic difference case.

One objective: Given any Poisson projective surface, associate noncommutative deformation parametrized by $J(C_\alpha)$ (e.g., when C_α is nodal, get $q \in k^*$). (Also find difference/differential interpretation(s) and understand derived equivalences between them!)

The case C_α smooth is the most straightforward (the geometry is simplest), but most results hold in general (+ for general elliptic difference equations, etc.).*

*By contrast, nearly all work in special functions corresponds to C_α reducible or even nonreduced!

The Higgs bundle setting suggests that the right thing to look at originally are noncommutative ruled surfaces. Van den Bergh gives a general construction of such things, as well as a construction of blowups. There are some issues, though:

Problem 1: We need to relate noncommutative ruled surfaces to difference/differential operators.

Problem 2: Very little is known about the birational geometry of such surfaces; even the fact that blowups in sufficiently distinct points commute is not known!

Noncommutative ruled surfaces à la van den Bergh are difficult to classify in general, but one can show the only "truly" noncommutative cases are those deforming commutative surfaces. These are actually fairly straightforward to classify, and in each case* the category of "line bundles" on the surface has a representation in difference or differential operators (unique up to an overall scalar gauge transformation, which can essentially be fixed by choosing a sheaf corresponding to a first-order equation and gauging it to be trivial).

More precisely, such surfaces are classified by torsion-free sheaves on $C \times C$ with Chern class 2Δ. If the support is nonreduced, then there is a representation in differential operators; if both maps are separable, there is a representation in difference operators (which satisfy a symmetry condition if the support is integral).

*Insert usual characteristic 2 comment here

Main idea: given a scheme X, flat degree 2 morphisms $\pi_1 : X \to Y_1$, $\pi_2 : X \to Y_2$, there is a corresponding "double affine Hecke algebra" (the fiber coproduct of $\mathrm{End}_{Y_i}(\pi_{i*}\mathcal{O}_X)$ over $\mathcal{O}_X$) which can be represented by difference/differential reflection operators. When Y_i are smooth curves, this is Morita equivalent to its spherical algebra (which inherits a representation as symmetric difference or differential operators), and the Rees algebra w.r.to the filtration by the Bruhat order of D_∞ has Proj a noncommutative ruled surface.

If we twist by a line bundle on X, we can get every noncommutative ruled surface in this way. (This gives a representation in twisted operators, but can be gauged to a representation in untwisted operators.)

Transforms

An important deformation of $\mathbb{P}^1 \times \mathbb{P}^1$ comes from the following bigraded algebra: the space of operators of bidegree (m, n) is spanned by operators $x^i(\hbar D_x)^j$ for $0 \leq i \leq m$, $0 \leq j \leq n$. (In the limit $\hbar \to 0$, this becomes commutative.)

The operators $(-\hbar D_x)^i x^j$ satisfy the same relations*, giving an automorphism of this noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$ taking bidegree (m, n) to bidgree (n, m).

The analogous isomorphisms for other noncommutative deformations of $\mathbb{P}^1 \times \mathbb{P}^1$ also swap multiplication operators and differential/difference operators; gives 16 different generalized Fourier transforms (inc. Mellin, "middle convolution", elliptic), each a (formal) integral transform with explicit kernel.

*Take the Laplace transform!

Transforms, cont'd

There are also transformations corresponding to "elementary transformations" (blow up a point of a ruled surface, blow down its fiber); these typically correspond to scalar gauge transformations. There are also some "transformations" corresponding to the fact that commutative blowups in distinct points commute; these are trivial on functions, and just correspond to writing down the singularities in a different order! Note that if we blew up m points, the transformations act on $NS(X)$ or $K_0(X)$ as a reflection group of type E_{m+1}...

It's nontrivial to show that these various transformations actually make sense in the noncommutative setting; the difficulty is that there are far too many cases to consider explicitly (as in the Fourier case).

How to deal with this? It turns out that each construction (of ruled surfaces, of blowups, and of noncommutative $\mathbb{P}^2$) comes with a simple description of both the derived category* and the appropriate t-structure. So we can construct isomorphisms between noncommutative surfaces by constructing derived equivalences and checking that they preserve the t-structure. This turns out to be easy and mostly independent of which case we are in. (There are a few cases in which one of the atomic transformations fails, but these can be completely understood.)

*In the differential graded sense.

Isomonodromy

One can show that twisting a sheaf by a line bundle acts as a gauge transformation/isomonodromy deformation, and such a transformation which does not introduce apparent singularities arises by twisting. So discrete isomonodromy deformations are *intrinsic* to the geometry. We can thus apply the various birational transformations to obtain different interpretations of our nonautonomous integrable systems as isomonodromy transformations. This can change the qualitative structure of the linear equation: e.g., Painlevé VI also has an interpretation in terms of symmetric difference equations with a certain type of singularity at infinity.

Other (non-derived) results

The standard GIT construction of the semistable moduli space of sheaves on a commutative surface requires certain inequalities that actually fail as written in the noncommutative case. It is still open in general to show that these are projective, with two exceptions: sheaves of rank 1 ($\sim$ Hilbert schemes), and sheaves of rank 0. (The surface X_0 above is $X^{[1]}$.) For $n > 1$, this gives rise to versal deformation of the Hilbert scheme of any projective rational surface (with C_α smooth)...

There is also an analogue of the Riemann-Hilbert functor in the symmetric elliptic case, taking symmetric q-difference equations on $\mathbb{C}^*/\langle p\rangle$ to symmetric p-difference equations on $\mathbb{C}^*/\langle q\rangle$. (This recovers Birkhoff monodromy in a hand-waving limit as $p \to 0$.) This extends to an action of $\mathrm{SL}_3(\mathbb{Z})$...

Special to the rational case is that the moduli space can actually be 0-dimensional. The corresponding integrable system is boring, but the linear equations are still quite interesting: this includes a large variety of hypergeometric equations (${}_nF_{n-1}$, ${}_n\phi_{n-1}$, elliptic analogues, etc.). In geometric terms, these correspond to -2-curves, and correspond to (real) roots for the Kac-Moody Weyl group $W(E_{m+1})$.

More generally, given any Chern class on our surface, we can apply $W(E_{m+1})$ to try to make the equation simpler (put it in the fundamental chamber!). For each (even) dimension > 2, we can figure out the minimal representative of each $W(E_{m+1})$ orbit of Chern classes with moduli space of the given dimension.

Derived equivalences

The 2-dimensional case is of particular interest, as in that case the moduli space is itself a projective algebraic surface. If a 2-dimensional moduli space of sheaves on a (noncommutative) surface has a universal family, then it determines a map of derived categories. In the cases corresponding to *minimal* Lax pairs, this map of derived categories is an equivalence, and one can use this to identify the moduli space.

One finds that for any rational number d/r, and any type of second-order Lax pair for a (discrete) Painlevé equation, there is a corresponding Lax pair as a (discrete or continuous) connection on a vector bundle of rank $2r$ and degree d, in which all of the singularities occur with multiplicity r. (This includes the case $1/0 = \infty$, corresponding to Sakai's construction of discrete Painlevé equations)

Derived equivalences (cont'd)

The derived equivalences arising from moduli spaces are a special case of a more general construction: there is an action of the group $\mathbb{Z}.\Lambda^2_{E_8} \rtimes W(E_8) \times \mathsf{SL}_2(\mathbb{Z})$ on the moduli space of "noncommutative rational elliptic surfaces" such that two surfaces in the same orbit are always derived equivalent (with a specific identification of their Grothendieck groups). This reduces to a well-known action of the same group as derived equivalences in the family of (commutative) rational elliptic surfaces.

If the anticanonical curve is smooth, then these are the only pairs of derived equivalent noncommutative surfaces; this should be true in general, but requires us to understand derived equivalences between singular Gorenstein curves of genus 1.

Poisson structures

Another benefit of having a simple description of the derived category is that it lets us construct Poisson structures on very general moduli spaces. The most general result is that the "derived moduli stack" of objects in $D(X)$ has a natural "0-shifted" Poisson structure, and (derived) restriction to the anticanonical curve is Lagrangian (i.e., its fibers are symplectic leaves). This turns out not to be too hard; the hard part is showing that such a 0-shifted Poisson structure induces an actual Poisson structure on the algebraic space classifying "simple" objects (s.t. $\tau_{\leq 0} R\,\mathrm{End}(M) = k$), which we can show when there are no obstructions. (The analogue of a reduction due to Hurtubise/Markman extends this to general simple sheaves.)