Quantum affine algebras and Grassmannians

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Representations of quantum affine algebras

Denote $\mathfrak{g} = \mathfrak{sl}_n$, $I = [n-1] = \{1, \ldots, n-1\}$.

$U_q(\widehat{\mathfrak{g}})$ is the quantum affine algebra associated to $\mathfrak{g}$.

$\mathcal{P}$ = the free abelian group generated by $Y_{i,s}^{\pm 1}$, $i \in I$, $s \in \mathbb{Z}$.

$\mathcal{P}^+$ = the submonoid of $\mathcal{P}$ generated by $Y_{i,s}$, $i \in I$, $s \in \mathbb{Z}$.

Elements in $\mathcal{P}^+$ are called dominant monomials.

$\mathcal{P}_\ell^+$ = the submonoid of $\mathcal{P}^+$ generated by $Y_{i,i-2k-2}$, $i \in I$, $k \in [0, \ell]$.

Representations of quantum affine algebras

Hernandez and Leclerc in 2010 introduced a category $\mathcal{C}_\ell^{\mathfrak{g}}$ which is a subcategory of the category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules.

Simple finite dimensional $U_q(\widehat{\mathfrak{g}})$-modules in $\mathcal{C}_\ell^{\mathfrak{g}}$ are in one to one correspondence with elements in $\mathcal{P}_\ell^+$ (Chari-Pressley 1994).

Denote by $L(M)$ the simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module corresponding to $M \in \mathcal{P}^+$.

Cluster algebra structure on the Grothendieck ring of certain subcategory of the category of finite dimensional $U_q(\widehat{\mathfrak{g}})$-modules

Denote $X_{i,k}^{(s)} = Y_{i,s}Y_{i,s+2} \cdots Y_{i,s+2k-2}$. $L(X_{i,k}^{(s)})$ are called Kirillov-Reshetikhin modules. $L(X_{i,1}^{(s)})$ are called fundamental modules.

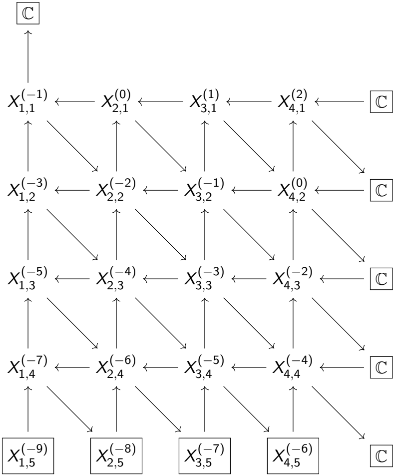
Denote $\mathcal{R}_\ell^{\mathfrak{g}} =$ Grothendieck ring of $\mathcal{C}_\ell^{\mathfrak{g}}$.

Theorem (Hernandez-Leclerc 2010)

The ring $\mathcal{R}_\ell^{\mathfrak{g}}$ has a cluster algebra structure. The cluster variables in the initial seed of the cluster algebra are certain Kirillov-Reshetikhin modules.

The initial cluster for $\mathcal{R}_4^{A_4}$

This is the initial cluster in the case of $U_q(\widehat{\mathfrak{sl}_5})$, $\ell = 4$.



Cluster algebra structure on Grassmannians

Scott in 2003 studied cluster algebra structures in coordinate rings of Grassmannians.

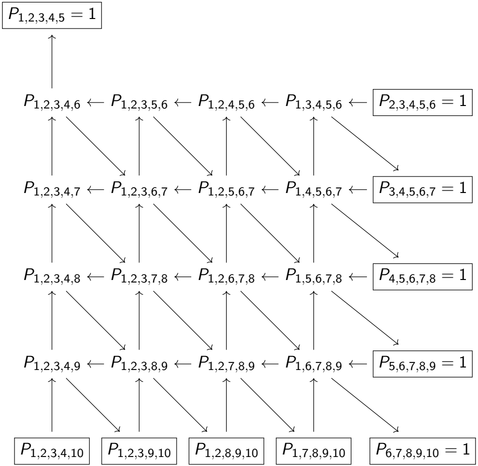
Theorem (Scott 2003)

The ring $\mathbb{C}[\mathrm{Gr}(n, m)]$ has a cluster algebra structure. The cluster variables in the initial seed are certain Plücker coordinates.

Denote $\mathbb{C}[\mathrm{Gr}(n, m, \sim)] = \mathbb{C}[\mathrm{Gr}(n, m)]/(P_{i,i+1,\ldots,i+n-1} - 1, i \in [m - n + 1])$.

The ring $\mathbb{C}[\mathrm{Gr}(n, m, \sim)]$ has a cluster algebra structure induced from the cluster algebra structure on $\mathbb{C}[\mathrm{Gr}(n, m)]$.

The initial cluster for $\mathbb{C}[\mathrm{Gr}(5, 10, \sim)]$



Isomorphism of the two cluster algebras

Denote $P^{(a,b,c)} = P_{j_1,\ldots,j_n}$, $j_1 = b$, $j_k = j_{k-1} - 1$, $k \in [2, a] \cup [a+2, n]$, $j_{a+1} - j_a = c$.

Theorem (Hernandez-Leclerc 2010)

The assignments $L(X_{i,t+1}^{(i-2t-2)}) \mapsto P^{(n-i+1,1,t+2)}$, $i \in I$, *extends to a ring isomorphism* $\Phi : \mathcal{R}_\ell^{A_{n-1}} \to \mathbb{C}[\mathrm{Gr}(n, n+\ell+1, \sim)]$.

Under the map Φ, Kirillov-Reshetikhin modules are sent to certain Plücker coordinates. A natural question is: what are the images of the simple modules in $\mathcal{R}_\ell^{A_{n-1}}$. To answer the question, we use rectangular tableaux with n rows.

Monoid $\mathrm{SSYT}(n, [n+\ell+1])$ of semi-standard Young tableaux

(1) $\mathrm{SSYT}(\mathrm{n}, [\mathrm{m}])$ = the set consisting of 1 (empty tableau) and semi-standard Young tableaux of rectangular shape with n rows and with entries in $[m]$.

(2) For $A, B \in \mathrm{SSYT}(\mathrm{n}, [\mathrm{m}])$, $A \cup B$ is the semi-standard tableau with n rows and the elements in the ith row are the union of elements in the ith row of A and B, $i \in [n]$.

Example

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 7 \\ \hline 6 & 8 \\ \hline \end{array} \cup \begin{array}{|c|c|}\hline 1 & 7 \\ \hline 2 & 9 \\ \hline 8 & 10 \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 7 \\ \hline 2 & 2 & 7 & 9 \\ \hline 6 & 8 & 8 & 10 \\ \hline \end{array}.$$

Monoid $\mathrm{SSYT}(n, [n+\ell+1])$ of semi-standard Young tableaux

(1) We say that $A \in \mathrm{SSYT(n, [m])}$ is a trivial tableau if either $A = 1$ or $A = \cup_j T_{i_j}$, where T_{i_j} is a one column tableau with entries $i_j, i_j + 1, \ldots, i_j + n - 1$, $i_j \in \mathbb{Z}_{\geq 1}$.

The tableau

3
4
5

is a trivial tableau.

(2) For $A \in \mathrm{SSYT}(n, [m])$, denote by $\widetilde{A} \subset A$ the semi-standard Young tableau with minimum number of columns such that $A = \widetilde{A} \cup A'$ for some trivial tableau A'.

Monoid $\mathrm{SSYT}(n, [n+\ell+1])$ of semi-standard Young tableaux

(1) For $A, B \in \mathrm{SSYT}(n, [m])$, define $A \sim B$ if either A, B are trivial tableaux or $\widetilde{A} = \widetilde{B}$.

$$\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 6 \\ \hline \end{array} \sim \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 3 & 4 \\ \hline 4 & 5 & 6 \\ \hline \end{array}.$$

(2) Denote $\mathrm{SSYT}(n, [m], \sim) = \mathrm{SSYT}(n, [m])/\sim$.

Lemma

$\mathrm{SSYT}(n, [m])$ and $\mathrm{SSYT}(n, [m], \sim)$ are commutative cancellative monoids under the multiplication "$\cup$".

Isomorphism of monoids $\mathcal{P}^+_{\ell,A_{n-1}} \to \mathrm{SSYT}(n,[n+\ell+1],\sim)$

Theorem (Chang-Duan-Fraser-L.)

The isomorphism $\Phi : \mathcal{R}_\ell^{A_{n-1}} \to \mathbb{C}[\mathrm{Gr}(n, n+\ell+1, \sim)]$ *induces an isomorphism of monoids* $\widetilde{\Phi} : \mathcal{P}^+_{\ell,A_{n-1}} \to \mathrm{SSYT}(n,[n+\ell+1],\sim)$.

$$\widetilde{\Phi}(Y_{1,-1}Y_{2,-4}Y_{1,-7}Y_{2,-6}Y_{1,-9}) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline 4 & 7 & 8 \\ \hline \end{array},$$

$$\widetilde{\Phi}(Y_{1,-1}Y_{1,-3}Y_{1,-5}Y_{2,-4}{Y_{1,-7}}^2Y_{2,-6}{Y_{1,-9}}^2) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline 7 & 8 & 8 \\ \hline \end{array}.$$

Fundamental modules correspond to certain Plücker coordinates

(1) For a Plücker coordinate P, denote by T_P the one-column tableau with entries from the indices of P.

(2) By T-systems, fundamental modules $L(Y_{i,s})$ corresponds to $T_{P_{(i,s)}}$, $P_{(i,s)} = P^{(n-i,\frac{i-s}{2},2)}$.

Fundamental modules correspond to certain Plücker coordinates

$$[Y_{1,-1}][Y_{1,-3}] = [Y_{1,-3}Y_{1,-1}] + [Y_{2,-2}],$$
$$P_{124}P_{235} = P_{125}P_{234} + P_{123}P_{245}.$$

Note that we set $P_{123} = 1$, $P_{234} = 1$.

$$\widetilde{\Phi}(Y_{1,-1}) = \boxed{\begin{matrix}1\\2\\4\end{matrix}}, \widetilde{\Phi}(Y_{1,-3}) = \boxed{\begin{matrix}2\\3\\5\end{matrix}}, \widetilde{\Phi}(Y_{1,-3}Y_{1,-1}) = \boxed{\begin{matrix}1\\2\\5\end{matrix}}, \widetilde{\Phi}(Y_{2,-2}) = \boxed{\begin{matrix}2\\4\\5\end{matrix}}.$$

From dominant monomials to tableaux

Denote $T_M = \widetilde{\Phi}(M)$ and $M_T = \widetilde{\Phi}^{-1}(T)$.
Let $M = Y_{2,0}Y_{1,-3}Y_{2,-2}Y_{1,-5}$. Then

$$T_M = \widetilde{\Phi}(Y_{2,0}) \cup \widetilde{\Phi}(Y_{1,-3}) \cup \widetilde{\Phi}(Y_{2,-2}) \cup \widetilde{\Phi}(Y_{1,-5})$$

$$= \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \cup \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 5 \\ \hline \end{array} \cup \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline 5 \\ \hline \end{array} \cup \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline 6 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}.$$

From tableaux to dominant monomials

Let

$$T = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 6 \\ \hline \end{array} = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \cup \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 5 \\ \hline \end{array} \cup \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline 6 \\ \hline \end{array}.$$

The unique multi-set of Plücker coordinates $P^{(a_i,b_i,2)}$, $i \in [k]$, $k \in \mathbb{Z}_{\geq 1}$ such that $T = \cup_{i=1}^{k} T_{P^{(a_i,b_i,2)}}$ is

$$\begin{aligned}\{P_{1,3,4}, P_{2,3,5}, P_{3,4,6}\} &= \{P^{(1,1,2)}, P^{(2,2,2)}, P^{(2,3,2)}\} \\ &= \{P_{(2,0)}, P_{(1,-3)}, P_{(1,-5)}\}.\end{aligned}$$

We have $M_T = \widetilde{\Phi}^{-1}\left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 6 \\ \hline \end{array}\right) = Y_{2,0} Y_{1,-3} Y_{1,-5}$.

Cluster monomials in a Grassmannian cluster algebra

Recall that $T_M = \widetilde{\Phi}(M)$ and $M_T = \widetilde{\Phi}^{-1}(T)$.

Definition

For a semi-standard tableau $T \in \mathrm{SSYT}(n, [n+\ell+1], \sim)$, $n \in \mathbb{Z}_{\geq 2}$, $\ell \in \mathbb{Z}_{\geq 1}$, we define $\mathrm{ch}(T) \in \mathbb{C}[\mathrm{Gr}(n, n+\ell+1, \sim)]$ by $\mathrm{ch}(T) = \Phi(L(M_T))$.

Corollary

The isomorphism $\Phi : \mathcal{R}_\ell^{A_{n-1}} \to \mathbb{C}[\mathrm{Gr}(n, n+\ell+1, \sim)]$ *sends a module* $L(M)$ *to* $\mathrm{ch}(T_M)$ *and* $\Phi^{-1}(\mathrm{ch}(T)) = L(M_T)$.

Cluster monomials in a Grassmannian cluster algebra

(1) A simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module is called prime if it is not isomorphic to a tensor product of two non-trivial modules.

(2) A simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module M is called real if $M \otimes M$ is simple.

Theorem (Qin 2017, Kashiwara-Kim-Oh-Park 2019)

Every cluster monomial (resp. cluster variable) in $\mathcal{R}_\ell^{\mathfrak{g}}$ corresponds to the isomorphism class of a real (resp. real prime) simple object in $\mathcal{C}_\ell^{\mathfrak{g}}$.

Cluster monomials in a Grassmannian cluster algebra

We call T prime (resp. real) if $L(M_T)$ is prime (resp. real).

Theorem

Every cluster monomial (resp. cluster variable) in $\mathbb{C}[\mathrm{Gr}(n, m, \sim)]$, $n < m$, is of the form $\mathrm{ch}(T)$ *for some real tableau (resp. prime real tableau) $T \in \mathrm{SSYT}(n, [m], \sim)$.*

A natural question is: how to compute $\mathrm{ch}(T)$. To answer the question, we use Arakawa-Suzuki's formula.

Arakawa-Suzuki's formula

F is a non-archimedean local field with a normalized absolute value $|\cdot|$.

For any reductive group G over F, let $\mathcal{C}(G)$ be the category of complex, smooth representations of $G(F)$ of finite length and let $\mathrm{Irr}G$ be the set of irreducible objects of $\mathcal{C}(G)$ up to equivalence.

$G_n = GL_n$.

For $\pi_i \in \mathcal{C}(G_{n_i})$, $i = 1, 2$, $\pi_1 \times \pi_2 \in \mathcal{C}(G_{n_1+n_2})$ is the representation which is parabolically induced from $\pi_1 \otimes \pi_2$.

Irr_c is the set of supercuspidal representations of G_n, $n > 0$.

Arakawa-Suzuki's formula

A segment is a finite non-empty subset of Irr_c of the form $\Delta = \{\rho_1, \ldots, \rho_k\}$, where $\rho_{i+1} = \rho_i \nu$, $i \in [k-1]$, where ν is the character $\nu(g) = |\det(g)|$.

We fix $\rho \in \mathrm{Irr}_c$ and write a segment $\{\rho\nu^i : i \in [a, b]\}$ as $[a, b]$, $a, b \in \mathbb{Z}$, $a \leq b$.

A multi-segment is a formal finite sum $\mathbf{m} = \sum_{i=1}^{k} \Delta_i$ of segments.

For $\Delta = \{\rho_1, \ldots, \rho_k\}$, $\mathrm{Z}(\Delta) = \mathrm{soc}(\rho_1 \times \cdots \times \rho_k)$, where $\mathrm{soc}(\pi)$ denotes the socle of π (the largest semisimple subrepresentation of π).

For a multi-segment $\mathbf{m} = \sum_{i=1}^{k} \Delta_i$,
$\zeta(\mathbf{m}) = \mathrm{Z}(\Delta_1) \times \cdots \times \mathrm{Z}(\Delta_k)$, $\mathrm{Z}(\mathbf{m}) = \mathrm{soc}(\zeta(\mathbf{m}))$.

Arakawa-Suzuki's formula

For $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k$, denote by S_λ the subgroup of S_k consisting of elements σ such that $\lambda_{\sigma(i)} = \lambda_i$.
For $\mu = (\mu_1, \ldots, \mu_k)$, $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k$, let $\mathbf{m}_{\mu,\lambda} = \sum_{i=1}^k [\mu_i, \lambda_i]$.

Theorem (Arakawa-Suzuki 1998, see also Lapid-Minguez 2018)

For $w \in S_n$ which is of maximal length in $S_\lambda w S_\mu$,

$$[\mathrm{Z}(\mathbf{m}_{w\mu,\lambda})] = \sum_{w' \in S_k} (-1)^{\ell(w'w)} p_{w'w_0, ww_0}(1)[\zeta(\mathbf{m}_{w'\mu,\lambda})],$$

where $p_{y,y'}(q)$ $(y, y' \in S_k)$ is the Kazhdan-Lusztig polynomial.

Equivalence of categories

Let H_N ($N \in \mathbb{Z}_{\geq 1}$) be the Iwahori-Hecke algebra of $GL_N(F)$.

(1) The category of finite-dimensional representations of H_N is equivalent to the category of smooth finite-length representations of $GL_N(F)$ which are generated by the vectors which are fixed under the Iwahori subgroup.

(2) Chari and Pressley in 1996 proved that when $N \leq n$, there is an equivalence between the category of finite dimensional representations of H_N and the subcategory of finite dimensional representations of $U_q(\widehat{\mathfrak{sl}_n})$ consisting of those representations whose irreducible components under $U_q(\mathfrak{sl}_n)$ all occur in the N-fold tensor product of the natural representation of $U_q(\mathfrak{sl}_n)$.

Monoid of multi-segments and monomial of dominant monomials

Consider all groups $GL_n(F)$, $n \geq 0$ at once and denote by Irr the set of equivalence classes of irreducible representations of $GL_n(F)$, $n \geq 0$.
By the Zelevinsky classification, Irr is in one-to-one correspondence with the monoid of multisegments.
There is an isomorphism of monoids (Chari-Pressley 1996):

$$\text{monoid of multi-segments} \to \mathcal{P}^+$$
$$[a, b] \mapsto Y_{b-a+1,a+b-1}$$

Denote by $M_{\mathbf{m}}$ the dominant monomial corresponding to $\mathbf{m}$ and $\mathbf{m}_M$ the multi-segment corresponding to M.

Dominant monomials and multi-segments

Let

$$M = Y_{2,0}Y_{1,-3}Y_{2,-2}Y_{1,-5}Y_{2,-6}Y_{2,-8}.$$

Then

$$\mathbf{m}_M = [0,1] + [-1,0] + [-1,-1] + [-2,-2] + [-3,-2] + [-4,-3].$$

Segments, fundamental modules, certain one-column tableaux

For a segment $[a, b]$, we denote

$$M_{[a,b]} = \begin{cases} Y_{b-a+1,a+b-1}, & a < b+1, \\ 1, & a = b+1, \\ 0, & a > b+1. \end{cases}$$

We use the convention that $\mathrm{ch}(0) = 0$ and $\mathrm{ch}(1) = 1$.
For a pair of k-tuples $(\mu, \lambda) \in \mathbb{Z}^k \times \mathbb{Z}^k$, we define multi-sets

$$\mathrm{Fund}_M(\mu, \lambda) = \{M_{[\mu_i,\lambda_i]} : i \in [k]\}.$$

Relation between a segment and a one-column tableau:
$[a, b] \mapsto T_{[1-a,1-a+n]\setminus\{n-b\}}$.

For $M \in \mathcal{P}_\ell^+$, there is a unique $k = k_M \in \mathbb{Z}_{\geq 1}$, a unique $w_M \subset S_k$, and a unique pair $(\mu, \lambda) = (\mu_M, \lambda_M) \in \mathbb{Z}^k \times \mathbb{Z}^k$, $\mu_1 \geq \cdots \geq \mu_k$, $\lambda_1 \geq \cdots \geq \lambda_k$, such that the multi-segment $\mathbf{m}_M$ is $\mathbf{m}_{w_M\mu,\lambda}$ and w_M is of maximal length in $S_{\lambda_M} w_M S_{\mu_M}$.

q-character formula

Translating Arakawa-Suzuki's formula to the language of q-characters, we have
For a simple $U_q(\widehat{\mathfrak{sl}_n})$-module $L(M)$,

$$\chi_q(L(M)) = \sum_{w' \in S_k} (-1)^{\ell(w' w_M)} p_{w' w_0, w_M w_0}(1) \prod_{M' \in \mathrm{Fund}_M(w' \mu_M, \lambda_M)} \chi_q(L(M'))$$

where $k = k_M$, w_0 is the longest word in S_k, $p_{u,v}(t)$ is the Kazhdan-Lusztig polynomial.

A formula for $\mathrm{ch}(T)$

Suppose that T' has columns $T'_1, \ldots, T'_k$. Each column T'_i has content of the form $[a_i, a_i + n] \setminus \{c_i\}$, with $c_i \in [a_i, a_i + n]$ (we say that T' has small gaps).

For a permutation $w \in S_k$, we define $w \cdot T'$ in two cases. First, suppose that $c_i \in [a_{w(i)}, a_{w(i)} + n]$ for all i, then we define $w \cdot T'$ be the column-increasing tableaux whose ith column is $[a_{w(i)}, a_{w(i)} + n] \setminus \{c_i\}$.

Second, if $c_i \notin [a_{w(i)}, a_{w(i)} + n]$ for some i, then we say that $w \cdot T'$ is undefined, and we define $P_{w \cdot T'} := 0 \in \mathbb{C}[\mathrm{Gr}(n, m)]$.

A formula for $\mathrm{ch}(T)$

Theorem

Let T be any tableaux and T' the tableau in its equivalence class with small gaps. Then

$$\mathrm{ch}(T) = \sum_{w' \in S_k} (-1)^{\ell(w'w)} p_{w'w_0, ww_0}(1) P_{w' \cdot T'} \in \mathbb{C}[\mathrm{Gr}(n, m, \sim)]$$

with $w_0 \in S_k$ the longest element and $w = w_{M_T}$.

$$\mathrm{ch}\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline\end{array}\right) = P_{124}P_{356} - P_{123}P_{456}. \qquad (1)$$

Examples of the q-character formula

Let $M = Y_{1,-5}Y_{1,-3}Y_{2,-2}Y_{2,0}$. Then

$$\begin{aligned}\chi_q(L(M)) = {} & \chi_q(Y_{2,-2})\chi_q(Y_{4,-2}) - \chi_q(Y_{3,-1})\chi_q(Y_{3,-3}) \\ & + \chi_q(Y_{1,-1})\chi_q(Y_{3,-1})\chi_q(Y_{2,-4}) \\ & - \chi_q(Y_{2,0})\chi_q(Y_{2,-2})\chi_q(Y_{2,-4}) \\ & - \chi_q(Y_{1,-1})\chi_q(Y_{1,-3})\chi_q(Y_{3,-1})\chi_q(Y_{1,-5}) \\ & + \chi_q(Y_{2,0})\chi_q(Y_{1,-3})\chi_q(Y_{2,-2})\chi_q(Y_{1,-5}).\end{aligned}$$

Examples of the q-character formula

In the above q-character formula, $Y_{i,s}$ are identified with 1 for $i = n$ and identified with 0 for $i \geq n+1$.
In the case of $\mathfrak{g} = \mathfrak{sl}_3$, we have

$$\begin{aligned}\chi_q(L(M)) = &-1 + \chi_q(Y_{1,-1})\chi_q(Y_{2,-4}) - \chi_q(Y_{2,0})\chi_q(Y_{2,-2})\chi_q(Y_{2,-4}) \\ &- \chi_q(Y_{1,-1})\chi_q(Y_{1,-3})\chi_q(Y_{1,-5}) \\ &+ \chi_q(Y_{2,0})\chi_q(Y_{1,-3})\chi_q(Y_{2,-2})\chi_q(Y_{1,-5}).\end{aligned}$$

Examples of the q-character formula

In the language of Grassmannian, this formula is

$$\mathrm{ch}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline\end{array}\right) = -1 + P_{124}P_{356} - P_{134}P_{245}P_{356} - P_{124}P_{235}P_{346} + P_{134}P_{235}P_{245}P_{346}.$$

Using Plücker relations, we can write this formula in terms of semi-standard Young tableaux.

Examples of the q-character formula

Using Plücker relations and $P_{123} = P_{234} = P_{345} = P_{456} = 1$, we have

$$\begin{aligned}
&P_{135}P_{246} - P_{125}P_{346} - P_{134}P_{256} + P_{124}P_{356} - 2P_{123}P_{456} \\
&= (P_{235}P_{134} - P_{123}P_{345})(P_{346}P_{245} - P_{234}P_{456}) \\
&\quad - (P_{124}P_{235} - P_{123}P_{245})P_{346} \\
&\quad - P_{134}(P_{245}P_{356} - P_{235}P_{456}) + P_{124}P_{356} - 2 \\
&= (P_{235}P_{134} - 1)(P_{346}P_{245} - 1) - (P_{124}P_{235} - P_{245})P_{346} \\
&\quad - P_{134}(P_{245}P_{356} - P_{235}) + P_{124}P_{356} - 2 \\
&= -1 + P_{124}P_{356} - P_{134}P_{245}P_{356} - P_{124}P_{235}P_{346} + P_{134}P_{235}P_{245}P_{346} \\
&= \mathrm{ch}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}\right).
\end{aligned}$$

Examples of the q-character formula

$$\mathrm{ch}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}\right) = P_{135}P_{246} - P_{125}P_{346} - P_{134}P_{256} + P_{124}P_{356} - 2P_{123}P_{456}$$

$$= P_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}} - P_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array}} - P_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array}} + P_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array}} - 2P_{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array}}$$

Recall that for a semi-standard Young tableaux T, we denote $P_T = P_{T_1} \cdots P_{T_m}$, where T_i's are columns of T, P_{T_i} is the Plücker coordinate with entries from a one-column tableau T_i.

Real modules and non-real modules

We call a semi-standard Young tableau T real if the corresponding module $L(M_T)$ is real.

T is real if and only if $\mathrm{ch}(T)\mathrm{ch}(T) = \mathrm{ch}(T \cup T)$.

Let $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}$. Then $\mathrm{ch}(T)$ is a cluster variable and T is real.

The following are smallest prime non-real tableaux for $\mathrm{Gr}(3, 9)$ and $\mathrm{Gr}(4, 8)$.

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & 7 \\ \hline 5 & 8 & 9 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 8 \\ \hline 6 & 7 & 9 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 5 & 8 & 9 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 7 \\ \hline 6 & 8 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline 7 & 8 \\ \hline \end{array}.$$

Thanks for your attention.
Happy birthday to Prof. Vitaly Tarasov and Prof. Alexander Varchenko!