GOOD COMPACTIFICATION THEOREM AND THE RING OF CONDITIONS OF $(\mathbb{C}^*)^n$

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INTRODUCTION

The **ring of conditions** $\mathcal{R}_n$ was introduced by De Concini and Procesi in 1980-th. It is a version of intersection theory for algebraic cycles in $(\mathbb{C}^*)^n$ (actually they introduced an analogues ring for any symmetric space). De Concini and Procesi reduced basically the ring $\mathcal{R}_n$ to the cohomology rings of smooth toric varieties using the **good compactification theorem**.

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I will present a new elementary proof of the good compactification theorem and will discuss two nice geometric descriptions of $\mathcal{R}_n$.

Tropical geometry provides the first description.

The second one can be formulated in terms of **volume function** on the cone of convex polyhedra with integral vertices in $\mathbb{R}^n$.

1. THE RING OF CONDITIONS $\mathcal{R}_n$.

Two k-dimensional cycles $X_1, X_2 \subset (\mathbb{C}^*)^n$ are *equivalent* $X_1 \sim X_2$ if for any $(n-k)$-dimensional cycle $Y \subset (\mathbb{C}^*)^n$ and for almost any $g \in (\mathbb{C}^*)^n$ we have $< X_1, gY > = < X_2, gY >$ (here $< A, B >$ is the intersection index of A and B).

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If $X_1 \sim X_2$ and $Y_1 \sim Y_2$ then for almost any $g_1, g_2 \in (\mathbb{C}^*)^n$ we have $X_1 \cap g_1 Y_1 \sim X_2 \cap g_2 Y_2$.

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One can defined the product $X * Y$ of equivalence classes X and Y as the equivalence classes of the intersection $X_1 \cap g_1 Y_1$ where X_1 and Y_1 are representatives of X and Y.

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The ring of conditions $\mathcal{R}_n$ is the ring of the equivalence classes of algebraic cycles with the multiplication $*$ and with the tautological addition.

2. GOOD COMPACTIFICATION

A complete toric variety $M \supset (\mathbb{C}^*)^n$ is a *good compactification* for a k-dimensional algebraic variety $X \subset (\mathbb{C}^*)^n$ if the closure of X in M does not intersect orbits of the action of $(\mathbb{C}^*)^n$ on M whose codimension is bigger than k.

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One can proof theorem 1 using the **universal Grobner basis** technic. Let us discuss its elementary proof.

3. BERGMAN SET OF $X \subset (\mathbb{C}^*)^n$

A vector $k \in \mathbb{Z}^n$ is **essential** for X if there is a meromorphic map $f : (\mathbb{C}, 0) \to X \subset (\mathbb{C}^*)^n$ where $f(t) = ct^k + \dots$ and $c \in (\mathbb{C}^*)^n$. A ray is **essential** for X if it contains an essential vector.

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A finite union of rational cones $\sigma_i \subset \mathbb{R}^n$ is the **Bergman set** $B(X)$ of X iff its set of essential rays is the set of a rational rays in $B(X)$.

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Theorem 3

Any variety $X \subset {}^*)^n$ has the (unique) Bergman set $B(X)$. If each irreducible component of X has complex dimension m then $B(X)$ is a finite union of rational cones σ_i with $\dim_{\mathbb{R}} \sigma_i = m$.

Theorem 3 is equivalent to the good compactification theorem.

4. RING $\mathcal{R}_n$ AND COHOMOLOGY RING

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For a complete smooth toric variety $M^n \supset (\mathbb{C}^*)^n$ and for any k -dimensional cycle $X = \sum k_i X_i$ one can defined the cycle $\overline{X}$ in M^n as $\sum k_i \overline{X}_i$ where $\overline{X}_i$ is the closure in M^n of $X_i \subset (\mathbb{C}^*)^n$.

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The cycle $\overline{X}$ defines an element $\rho(\overline{X})$ in $H^{2(n-k)}(M^n, \Lambda)$ whose value on the closure $\overline{O}_i$ of an $(n-k)$-dimensional orbit O_i in M^n is equal to the intersection index $< \overline{X}, \overline{O}_i >$.

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Theorem 4

If a smooth toric compactification M^n is good for cycles X, Y and Z where $Z = X * Y$, then the product $\rho(X)\rho(Y)$ in the cohomology ring $H^*(M^n, \Lambda)$ of the elements $\rho(X)$ and $\rho(Y)$ is equal to $\rho(Z)$.

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5.1. Ring encoded by a polynomial P

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Let $D(\mathcal{L})$ be the ring of linear differential operators on $\mathcal{L}$ with constant coefficients. This ring is graded by the order of the operators. It is generated by Lie derivatives L_v along constant vector fields $v(x) \equiv v \in \mathcal{L}$ and by operators of multiplication on complex constants.

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The ring $D(\mathcal{L})$ is graded by the order of the operators:
$D(\mathcal{L}) = D_0 \oplus D_1 \oplus \ldots.$

5.2. Ring encoded by P, continuation

Let $I_P \subset D(\mathcal{L})$ be a set defined by the following condition: $L \in I_P \Leftrightarrow L(P) \equiv 0$. It is easy to see that I_P is a homogeneous ideal.

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One can to see that:
(1) $A(\mathcal{L}, P)$ is a graded ring with homogeneous components A_k where $0 \leq k \leq n = \deg P$;
(2) $A_0 = \mathbb{C}$;
3) there is a non-degenerate pairing between A_k and A_{n-k} with values in A_0, thus $A_k = A^*_{(n-k)}$ and $A_n \sim \mathbb{C}$.

5.3. Rings $H^*(M^n)$, $\mathcal{R}_n$ and the volume function

Let M^n be a smooth projective toric variety. Let L_n be the space of virtual convex polyhedra whose support functions are linear on each cone from the fan of M^n. Let $n!V$ be the degree n homogeneous polynomial on L_n whose value on $\Delta \in \mathcal{L}_n$ is the volume of Δ multiplied by n!.

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Theorem 7

The ring $\mathcal{R}_n$ is isomorphic to the ring $A(\mathcal{L}_n, n!V)$.

6.1. Λ-enriched fans

An **enriched** k-**fan** is a fan $\mathcal{F} \subset \mathbb{R}^n$ of some toric variety equipped with a **weight function** $c : \mathcal{F}_k \to \Lambda$ defined on the set $\mathcal{F}_k$ of all k-dimensional cones in $\mathcal{F}$. The **support** $|\mathcal{F}|$ of $\mathcal{F}$ is the union of all cones $|\sigma_i| \subset \mathbb{R}^n$ such that $\sigma_i \in \mathcal{F}_k$ and $c(\sigma_i) \neq 0$.

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Two enriched k-fans $\mathcal{F}_1$ and $\mathcal{F}_2$ are **equivalent** if:
1) their supports $|\mathcal{F}_1|$ and $|\mathcal{F}_2|$ are equal
2) their weight functions c_1 and c_2 induce the same weight function on every common subdivision of the fans $\mathcal{F}_1$ and $\mathcal{F}_2$.

6. TROPICALIZATION OF $\mathcal{R}_n(\Lambda)$

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Thus **an equivalence class** of enriched k-fans can be considered as **a linear combination of k-dimensional rational cones with nonzero coefficients in Λ defined up to subdivisions of cones**.

6.2: Balance condition for Λ-enriched fans

Let $\mathcal{F}$ be an enriched k-fan. For a cone $\sigma_i \in \mathcal{F}_k$ let $L_i^{\perp} \subset (\mathbb{R}^n)^*$ be the $(n-k)$-dimensional space dual to the span L_i of $\sigma_i \subset \mathbb{R}^n$. Let O be an orientation of σ_i. Denote by $e_i^{\perp}(O) \in \Lambda^{n-k} L_i^{\perp}$ the $(n-k)$-vector, such that:

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1) the integral volume of $|e_i^{\perp}(O)|$ in $L_i^{\perp}$ is equal to one;

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An enriched k-fan $\mathcal{F}$ satisfies **the balance condition** if for any orientation of any $(k-1)$-dimensional cone $\rho \in F_{k-1}$ the relation

$$\sum e_i^\perp(O(\rho))c(\sigma_i) = 0$$

holds, where c is the weight function and summation is taken over all $\sigma_i \in \mathcal{F}_k$ such that $\rho \subset \partial\sigma_i$ and $O(\rho)$ is such orientation of σ_i that the orientation of $\partial\sigma_i$ agrees with the orientation of ρ.

6.3: Intersection number of complementary fans

Let $\mathcal{F}_1$ and a $\mathcal{F}_2$ be balanced k-fan and $(n-k)$-fan. Cones $\sigma_i^1 \in \mathcal{F}_1$, $\sigma_j^2 \in \mathcal{F}_2$ with $\dim \sigma_i^1 = k$, $\dim \sigma_j^2 = n-k$ are a-**admissible** for a vector $a \in \mathbb{R}^n$ if $\sigma_i^1 \cap (\sigma_j^2 + a) \neq \emptyset$. Let $C_{i,j}$ be the index of $\Lambda_i \bigoplus \Lambda_j$ in $\mathbb{Z}^n$ where $\Lambda_i = L_i^1 \cap \mathbb{Z}^n$, $\Lambda_j = L_j^2 \cap \mathbb{Z}^n$ and L_i^1, L_j^2 are linear spaces spanned by σ_i^1, σ_j^2.

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Definition 8

The intersection number $c(0)$ of $\mathcal{F}_1$ and $\mathcal{F}_2$ is equal to $\sum C_{i,j} c_1(\sigma_i^1) c_2(\sigma_j^2)$, where $a \in \mathbb{R}^n$ is a generic vector and the sum is taken over all a-admissible couples σ_i^1, σ_j^2.

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Definition 9

The **tropical product** $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ is a 0-fan $\mathcal{F} = \{0\}$ with the weight $c(0)$ equal to the intersection number of the fans.

6.4: Ring $\mathcal{TR}_n(\Lambda)$ of balanced Λ-enriched fans

Consider a k-fan $\mathcal{F}_1$ and a m-fan $\mathcal{F}_2$ from the set $\mathcal{TR}_n(\Lambda)$ of all balanced Λ-enriched fans. Let d be $n-(k+m)$. If $d<0$ then $\mathcal{F}_1 \times \mathcal{F}_2 = 0$. If $d=0$ the fan $\mathcal{F}_1 \times \mathcal{F}_2$ is already defined. Below we define the d-fan $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ for $d>0$.

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Assume that $\mathcal{F}_1$ and and $\mathcal{F}_2$ are subfans of a complete fan $\mathcal{G}$. Then $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ also is a subfan of $\mathcal{G}$. The weight $c(\delta)$ of a con δ with $\dim \delta = d$ in $\mathcal{G}$ is defined below.

6.4: Ring $T\mathcal{R}_n(\Lambda)$ of balanced Λ-enriched fans

Consider a k-fan $\mathcal{F}_1$ and a m-fan $\mathcal{F}_2$ from the set $T\mathcal{R}_n(\Lambda)$ of all balanced Λ-enriched fans. Let d be $n-(k+m)$. If $d<0$ then $\mathcal{F}_1\times\mathcal{F}_2=0$. If $d=0$ the fan $\mathcal{F}_1\times\mathcal{F}_2$ is already defined. Below we define the d-fan $\mathcal{F}=\mathcal{F}_1\times\mathcal{F}_2$ for $d>0$.

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Let L be a space spanned by the cone δ and let $(\mathcal{F}_1)_\delta$ and $(\mathcal{F}_2)_\delta$ be the enriched subfans of $\mathcal{F}_1$ and of $\mathcal{F}_2$ consisting of all cones from these fans containing the cone δ.

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Consider a k-fan $\mathcal{F}_1$ and a m-fan $\mathcal{F}_2$ from the set $T\mathcal{R}_n(\Lambda)$ of all balanced Λ-enriched fans. Let d be $n-(k+m)$. If $d<0$ then $\mathcal{F}_1 \times \mathcal{F}_2 = 0$. If $d=0$ the fan $\mathcal{F}_1 \times \mathcal{F}_2$ is already defined. Below we define the d-fan $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ for $d>0$.

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Definition 10

The weight $c(\delta)$ of the cone δ in $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ is equal to the intersection number of the images under the factorization of $(\mathcal{F}_1)_\delta$ and $(\mathcal{F}_2)_\delta$ in the factor space $\mathbb{R}^n/L$.

6.5. Ring $\mathcal{R}_n(\Lambda)$ and homology of toric varieties

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Let $\Delta^{\perp}$ be the fan of a smooth complete projective toric variety M^n. Let $T\mathcal{R}_n(\Lambda, \Delta)$ be the ring of balanced Λ-enriched fans equal to Λ-linear combination of cones from the fan $\Delta^{\perp}$.

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Theorem 11

The ring $T\mathcal{R}_n(\Lambda, \Delta)$ is isomorphic to the intersection ring $H_*(M_\Delta, \Lambda)$. The component of $T\mathcal{R}_n(\Lambda, \Delta)$ consisting of k-fans under this isomorphism corresponds to the component $H_{2k}(M_\Delta, \Lambda)$.

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Theorem 12

The ring of conditions $\mathcal{R}_n(\Lambda)$ is isomorphic to the tropical ring $T\mathcal{R}_n(\Lambda)$ be the ring of balanced Λ-enriched fans.

CONGRATULATIONS TO

SASHA AND VITALY!