Extension quiver for Lie superalgebra $\mathfrak{q}(3)$

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ETH Zurich, August, 2019

Outline

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The queer Lie superalgebra

The *queer Lie superalgebra* $\mathfrak{q}(n)$ is defined as

$$\mathfrak{q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} : A, B \in \mathfrak{gl}_n(\mathbf{C}) \right\}.$$

- Consider the *Borel subsuperalgebra* $\mathfrak{b}$ defined by requiring $A, B \in \mathfrak{gl}_n(\mathbf{C})$ to be upper triangular.
- The *nilpotent subsuperalgebra* $\mathfrak{n} = [\mathfrak{b}_{\bar{0}}, \mathfrak{b}_{\bar{0}}] \oplus [\mathfrak{b}_{\bar{0}}, \mathfrak{b}_{\bar{1}}]$ consists of block matrices such that A, B are strictly upper triangular.
- The *Cartan subsuperalgebra* $\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$ consists of block matrices such that A, B are diagonal.

Note, $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ is not abelian since $[\mathfrak{h}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] \neq 0$.

Irreducible $\mathfrak{q}(n)$ Representations

- Simple representation of $\mathfrak{h}$ are Clifford modules.
- Define Verma module as $M(\lambda) := U(\mathfrak{q}(n)) \otimes_{U(\mathfrak{b})} v(\lambda)$.
- Then $L(\lambda)$ is (unique up to parity) simple quotient of Verma module.

The $\mathfrak{q}(n)$ dominant integral weights are

$$\Lambda^+ = \{(\lambda_1, \dots, \lambda_n) \in \mathfrak{h}_{\bar{0}}^* : \lambda_i - \lambda_{i+1} \in \mathbf{Z}_{\geq 0}, \lambda_i = \lambda_j \Rightarrow \lambda_i = \lambda_j = 0\}.$$

Theorem (Penkov-Serganova '86)

$L(\lambda) := M(\lambda)/N(\lambda)$ is finite dimensional if and only if $\lambda \in \Lambda^+$.

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The Category $\mathfrak{q}(n)$-mod

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There is a block decomposition

$$(\mathfrak{q}(n) - mod) = \bigoplus_{\chi: Z(\mathfrak{q}) \to \mathbf{C}} (\mathfrak{q}(n) - mod)_\chi.$$

Each $\lambda \in \mathfrak{h}_{\bar{0}}^*$, we may define χ_λ using the Harish-Chandra homomorphism. Define $\mathrm{wt}(\lambda) := \varepsilon_{\lambda_1} + \cdots + \varepsilon_{\lambda_n}$, where $\varepsilon_{\lambda_i} = -\varepsilon_{-\lambda_i}$.

Theorem (Sergeev '83)

For $\lambda, \mu \in \mathfrak{h}_{\bar{0}}^*$, $\chi_\lambda = \chi_\mu$ if and only if $wt(\lambda) = wt(\mu)$.

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Equivalence of Blocks

Theorem ([Ser-ICM] Theorem 5.8)

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda^+$ be a dominant integral weight. Then each block of $\mathfrak{q}(3)$-mod is equivalent to one of the following:

- *the strongly typical block $(\mathfrak{q}(1) - mod)_{(\lambda_1)}$, when $\lambda_i + \lambda_j \neq 0,\ \forall i, j$ and $\lambda_i \neq 0$.*
- *the typical block $(\mathfrak{q}(1) - mod)_{(0)}$, when $\lambda_i + \lambda_j \neq 0,\ \forall i, j$ and some $\lambda_k = 0$.*
- *the "half-standard" block $(\mathfrak{q}(3) - mod)_{(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2})}$ when $\lambda_i + \lambda_j = 0$ and λ_k nonzero half-integer for i, j, k distinct.*
- *the standard block $(\mathfrak{q}(3) - mod)_{(1,0,0)}$ when $\lambda_i + \lambda_j = 0$ and λ_k nonzero integer for i, j, k distinct.*
- *the principal block $(\mathfrak{q}(3) - mod)_{(0,0,0)}$ when $\lambda = (\lambda_1, 0, -\lambda_1)$.*

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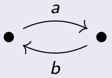
$\mathfrak{q}(3)$ Ext quiver

Theorem (G., Serganova '19)

Every block of $\mathfrak{q}(3)$-mod is equivalent to the category of finite-dimensional modules over one of the following algebras given by quiver and relations:

- *(Strongly typical block)*

- *(Typical block)*

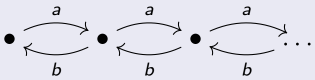


with relations $ab = ba = 0$.

$\mathfrak{q}(3)$ Ext quiver

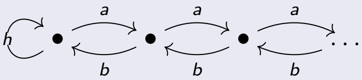
Theorem

- *("half-standard" block)*



with relations $a^2 = b^2 = 0, ab = ba.$

- *(Standard block)*

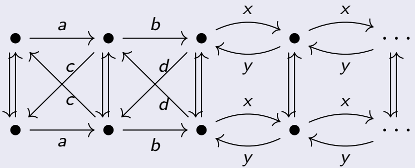


with relations
$a^2 = b^2 = 0, ab = ba, bah = hba, h^2 = 0, bh = ha = ab = 0.$

$\mathfrak{q}(3)$ Ext quiver

Theorem

- *(The principal block)*



With relations

$$x^2 = y^2 = 0, \quad xb = bd = ca = 0$$
$$xy = yx, \quad yx = bacd \quad dbac = acdb$$
$$\theta^2 = 0, \theta\gamma = \gamma\theta \text{ for } \gamma \in \{a, b, c, d, x, y\}.$$

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Standard and Principal Block

Let $t >> 0, a, b > 2$. Then

$$\begin{aligned}\mathrm{Ext}_{\mathfrak{q}(3)}(L(a,1,-a), L(b,1,-b)) &= \mathrm{Ext}_{\mathfrak{q}(3)}(L(t,a,-a), L(t,b,-b)) \\ &= \mathrm{Ext}_{\mathfrak{q}(1)\times\mathfrak{q}(2)}(L(t)\boxtimes L(a,-a), \\ &\qquad L(t)\boxtimes L(b,-b)) \\ &= \mathrm{Ext}_{\mathfrak{q}(2)}(L(a,-a), L(b,-b))\end{aligned}$$

Then make explicit computation showing $\mathrm{Ext}_{\mathfrak{q}(3)}(L(1,0,0), L(a,1,-a)) = \mathbf{C}$ if $a = 2$, and 0 if $a > 2$.

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Lemma (Translation Functors)

Let $a \geq 2$. Then

$$TL(a,0,-a) = L(a,1,-a)$$

$$T^*L(a,1,-a) = \frac{L(a,0,-a)}{\Pi L(a,0,-a)} \oplus \frac{\Pi L(a,0,-a)}{L(a,0,-a)}.$$

where $T(V) := (V \otimes L(1,0,0))_{\chi_{(1,0,0)}}$.

Then make explicit computations for $\mathrm{Ext}_{\mathfrak{q}(3)}(\mathbf{C}, L(a,0,-a))$ for $a \leq 2$.

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BGG Reciprocity

We define the *Euler characteristic* as

$$\mathcal{E}(\lambda) := \sum_{\mu} \sum_{i=0}^{\dim (G/B)_{\bar{0}}} (-1)^i [\Gamma_i(G/B, \mathcal{C}_\lambda) : L(\mu)][L(\mu)],$$

where Γ_i is a Zuckerman functor.

Theorem (G., Serganova '19)

Suppose $\mathcal{E}(\mu) = \sum_\lambda b(\mu,\lambda)[L(\lambda)]$*. Then there exists coefficients* $a_{\lambda,\mu}$ *such that*

$$[P(\lambda)] = \sum_{\mu \in \Lambda_0^+} a(\lambda,\mu)\mathcal{E}(\mu).$$

and

$$a(\lambda,\mu) = \begin{cases} \frac{2^{\dim \mathfrak{h}_{\bar{1}}}}{(\dim \mathcal{C}_\mu)^2} b(\mu,\lambda) & \text{if } L(\lambda) \text{ Type M} \\ \frac{2^{1+\dim \mathfrak{h}_{\bar{1}}}}{(\dim \mathcal{C}_\mu)^2} b(\mu,\lambda) & \text{if } L(\lambda) \text{ Type Q} \end{cases}$$

As a corollary, we compute the radical filtrations of $P(\lambda)$ for all λ.

References I

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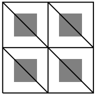
Self Extensions

Let $\lambda = (\lambda_1, ..\lambda_k, 0, .., 0, -\lambda_{k+m+1}, .., -\lambda_n)$ such that all $\lambda_i > 0$ be a dominant integral weight in $\mathfrak{q}(n)$. Consider the parabolic $\mathfrak{p}$ whose Levi subalgebra is

$\mathfrak{l} =$  .

Proof sketch:

$$\begin{aligned}\mathrm{Ext}_{\mathfrak{q}(n)}(L(\lambda), \Pi L(\lambda)) &= \mathrm{Ext}_{\mathfrak{l}}(L(\lambda)^{n_{\mathfrak{p}}}, \Pi L(\lambda)^{\mathfrak{n}_{\mathfrak{p}}}) \\ &= \mathrm{Ext}_{\mathfrak{q}(m)}(\mathbf{C}, \Pi\mathbf{C}) \\ &= \begin{cases}\mathbf{C} \text{ if } m > 0 \\ 0 \text{ else.}\end{cases}\end{aligned}$$