A Verlinde formula for twisted conformal blocks

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Twisted affine Lie algebras

- Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and Γ a finite group acting on $\mathfrak{g}$. For $\gamma \in \Gamma$, let $|\gamma|$ denote its order and consider the root of unity $\varepsilon = e^{\frac{2\pi i}{|\gamma|}}$. Let the element γ act on $\mathbb{C}((t))$ by $t \mapsto \varepsilon^{-1} t$.

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- The twisted affine Lie algebra is defined as

$$\widehat{L}(\mathfrak{g}, \gamma) := (\mathfrak{g} \otimes \mathbb{C}((t)))^{\gamma} \oplus \mathbb{C}c$$

with $c \in \widehat{L}(\mathfrak{g}, \gamma)$ central and Lie bracket given by

$$[X \otimes f, Y \otimes g] := [X, Y] \otimes fg + (X, Y)_{\mathfrak{g}} \cdot \mathrm{Res}_{t=0}(g \cdot df) \cdot c,$$

where $(\cdot, \cdot)_{\mathfrak{g}}$ is the normalized Killing form such that $(\theta^{\vee}, \theta^{\vee}) = 2$ for any long root θ of $\mathfrak{g}$.

Twisted affine Lie algebras

- Let $\widehat{L}(\mathfrak{g},\gamma)^{\geq 0} \subseteq \widehat{L}(\mathfrak{g},\gamma)$ be the Lie subalgebra

$$\widehat{L}(\mathfrak{g},\gamma)^{\geq 0} := (\mathfrak{g}\otimes\mathbb{C}[[t]])^{\gamma}\oplus\mathbb{C}c.$$

We have a decomposition

$$\widehat{L}(\mathfrak{g},\gamma)^{\geq 0} = \widehat{L}(\mathfrak{g},\gamma)^{+}\oplus\mathfrak{g}^{\gamma}\oplus\mathbb{C}c.$$

Integrable level ℓ representations

- Let us fix a positive integer ℓ called level. Let $\mathcal{C}^{\ell}(\mathfrak{g}, \gamma)$ be the category of level ℓ integrable representations of $\widehat{L}(\mathfrak{g}, \gamma)$, where level ℓ means that the central element c acts by the integer ℓ in the representation.

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- For each $\gamma \in \Gamma$, $\mathcal{C}^\ell(\mathfrak{g}, \gamma)$ is a finite semisimple abelian category with simple objects $\{\mathcal{H}_\lambda\}_{\lambda \in P^\ell(\mathfrak{g},\gamma)}$ parametrized by a certain finite subset $P^\ell(\mathfrak{g}, \gamma) \subseteq P_+(\mathfrak{g}^\gamma)$.

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- For $\lambda \in P^\ell(\mathfrak{g}, \gamma) \subseteq P_+(\mathfrak{g}^\gamma)$, let V_λ be the finite dimensional irreducible $\mathfrak{g}^\gamma$-rep of highest weight λ. We can define an action of $\widehat{L}(\mathfrak{g}, \gamma)^{\geq 0} = \widehat{L}(\mathfrak{g}, \gamma)^+ \oplus \mathfrak{g}^\gamma \oplus \mathbb{C}c$ on V_λ, where we let c act by ℓ and $\widehat{L}(\mathfrak{g}, \gamma)^+$ acts trivially.

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- Consider the induced module $\mathrm{Ind}_{\widehat{L}(\mathfrak{g},\gamma)^{\geq 0}}^{\widehat{L}(\mathfrak{g},\gamma)} V_\lambda$. It has a unique maximal submodule, and the quotient is $\mathcal{H}_\lambda$.

Integrable level ℓ representations

- For $\vec{\gamma} = (\gamma_1, \cdots, \gamma_n) \in \Gamma^n$ define the Lie algebra

$$\widehat{L}_n(\mathfrak{g}, \vec{\gamma}) := \oplus_i \widehat{L}(\mathfrak{g}, \gamma_i)/\mathfrak{z},$$

where $\mathfrak{z} = \{(a_1, \cdots, a_n) \subseteq \mathbb{C}c^{\oplus n} | \Sigma a_i = 0\}$.

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  where $\mathfrak{z} = \{(a_1, \cdots, a_n) \subseteq \mathbb{C}c^{\oplus n} | \Sigma a_i = 0\}$.
- If for $1 \leq i \leq n$, $\mathcal{H}_i \in \mathcal{C}^\ell(\mathfrak{g}, \gamma_i)$ then the tensor product $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ has a natural action of $\widehat{L}_n(\mathfrak{g}, \vec{\gamma})$.

Γ-covers of curves

- Let $\tilde{C} \to C$ be an admissible Γ-cover of (possibly nodal) complex projective curves with smooth marked points $\vec{p} = (p_1, \cdots, p_n)$ on C along with choice of lifts $\vec{\tilde{p}}$ on $\tilde{C}$ such that outside $\vec{p}$ and the nodes of C, the cover is a Γ-torsor.

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- In particular, we have an action of Γ on $\tilde{C}$ such that the stabilizer of any point is a cyclic group.
- The choice of the lifts $\tilde{p}_i$ of p_i and the orientation of complex curves determines a generator γ_i of the stabilizer of the point $\tilde{p}_i \in \tilde{C}$. Hence a Γ-cover as above determines $\vec{\gamma} \in \Gamma^n$ associated with the marked points.

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- In particular, we have an action of Γ on $\tilde{C}$ such that the stabilizer of any point is a cyclic group.
- The choice of the lifts $\tilde{p}_i$ of p_i and the orientation of complex curves determines a generator γ_i of the stabilizer of the point $\tilde{p}_i \in \tilde{C}$. Hence a Γ-cover as above determines $\vec{\gamma} \in \Gamma^n$ associated with the marked points.
- Let us also choose formal local parameters $\tilde{t}_i$ at $\tilde{p}_i$ such that $\tilde{t}_i^{|\gamma_i|}$ is a formal local parameter at p_i. This choice gives us an identification $\mathcal{K}_{\tilde{p}_i} \cong \mathbb{C}((t))$ which respects the γ_i action on both sides.

Twisted conformal blocks

- Let $(\tilde{C} \to C, \vec{\tilde{p}}, \vec{\tilde{t}})$ be an admissible Γ-cover as before. This determines $\vec{\gamma}$ and a Lie algebra homomorphism

$$\mathfrak{g}(\tilde{C} \setminus \Gamma \cdot \vec{\tilde{p}})^{\Gamma} \to \widehat{L}_n(\mathfrak{g}, \vec{\gamma}).$$

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- Hence given objects $\vec{\mathcal{H}} = (\mathcal{H}_1, \cdots, \mathcal{H}_n)$ such that each $\mathcal{H}_i \in \mathcal{C}^{\ell}(\mathfrak{g}, \gamma_i)$, we obtain an action of $\mathfrak{g}(\tilde{C} \setminus \Gamma \cdot \vec{\tilde{p}})^{\Gamma}$ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ and we consider the vector space of coinvariants:

$$\mathcal{V}_{\vec{\mathcal{H}},\Gamma}(\tilde{C} \to C, \vec{\tilde{p}}, \vec{\tilde{t}}) := [\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n]_{\mathfrak{g}(\tilde{C} \setminus \Gamma \cdot \vec{\tilde{p}})^{\Gamma}}.$$

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- If $\Gamma = \{1\}$, we obtain the well studied notion of usual conformal blocks. In this case Tsuchiya-Ueno-Yamada proved that these spaces satisfy axioms of a modular functor like propagation of vacua, factorization, flat projective connection (when you work over families of curves) etc.

Twisted conformal blocks

- Twisted conformal blocks have been studied by various authors like Frenkel-Szczesny, Damiolini, Hong-Kumar. In case Γ preserves a Borel subalgebra of $\mathfrak{g}$, Hong-Kumar proved propagation of vacua and factorization theorem and that the sheaves of conformal blocks are coherent and in fact locally free.

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- This naturally leads to the problem of determining the ranks of these bundles. In case $\Gamma = \{1\}$, the Verlinde formula answers this question.

Twisted Verlinde formula

Let $(C, \vec{p})$ be a smooth curve of genus g. The fundamental group of $C \setminus \vec{p}$ has a presentation of the form

$$\pi_1(C \setminus \vec{p}) = \langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \mu_1, \cdots, \mu_n | [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \mu_1 \cdots \mu_n = 1 \rangle.$$

Let us fix a group homomorphism $\chi : \pi_1(C \setminus \vec{p}, \star) \to \Gamma$ with image Γ°. Let $\gamma_i \in \Gamma$ be the image of μ_i. This determines an n-pointed admissible Γ-cover $(\tilde{C} \to C, \tilde{\vec{p}})$ such that all the lifts $\tilde{\vec{p}}$ lie in the same connected component of $\tilde{C}$ and the monodromy around the points $\tilde{\vec{p}}$ is given by $\vec{\gamma}$.

Twisted Verlinde formula

In joint work with S. Mukhopadhyay, we prove

Theorem

Suppose that Γ *preserves a Borel subalgebra in* $\mathfrak{g}$. *Then*

$$\dim \mathcal{V}_{\vec{\lambda},\Gamma}(\tilde{C} \to C, \vec{p}) = \sum_{\lambda \in P^{\ell}(\mathfrak{g})^{\Gamma^{\circ}}} \frac{S^{\gamma_1}_{\lambda_1,\lambda} \cdots S^{\gamma_n}_{\lambda_n,\lambda}}{\left(S_{0,\lambda}\right)^{n+2g-2}}.$$

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In the above formula, for any $\gamma \in \Gamma$, S^{γ} is a certain matrix known as the γ-crossed S-matrix which can be explicitly computed. The matrix S^{γ} is a $P^{\ell}(\mathfrak{g},\gamma) \times P^{\ell}(\mathfrak{g})^{\gamma}$-matrix. It is a square unitary matrix.

Γ-crossed modular fusion category

To prove the twisted Verlinde formula, we prove the following results

Theorem

Suppose that Γ preserves a Borel subalgebra in $\mathfrak{g}$. Then the twisted conformal blocks equip the Γ-graded finite semisimple category

$$\bigoplus_{\gamma\in\Gamma} \mathcal{C}^{\ell}(\mathfrak{g},\gamma)$$

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Theorem

There is a categorical twisted Verlinde formula that computes the fusion coefficients in any Γ-crossed modular fusion category in terms of crossed S-matrices.