EQUIVALENCES OF (CO)MODULE ALGEBRA STRUCTURES OVER HOPF ALGEBRAS

Representation theory and integrable systems - ETH, Zurich

Joint work with Alexey Gordienko and Joost Vercruysse

August 13, 2019

VRIJE UNIVERSITEIT BRUSSEL

INSTITUTUL DE MATEMATICA "SIMION STOILOW" AL ACADEMIEI ROMANE IMAR

STARTING POINT

Manin proved the existence of a universal coacting Hopf algebra $\text{aut}(A)$ on an algebra A:

- Manin, Y.I. – Quantum groups and non-commutative geometry, Centre de Recherches Mathématiques, Université de Montreal, 1988.
- Tambara, D. – The coendomorphism bialgebra of an algebra. *J. Fac. Sci. Univ. Tokyo Math.* **37** (1990), 425–456.

STARTING POINT

Equivalent Gradings:

STARTING POINT

Equivalent Gradings:

Definition

We say that gradings $\Gamma\colon A = \bigoplus_{g\in G} A^{(g)}$ and $\Gamma'\colon A = \bigoplus_{g'\in G'} A^{(g')}$ on an algebra A, are equivalent, if there exists an automorphism $\psi\colon A \to A$ of algebras such that for any $g \in \operatorname{supp}\Gamma^{a}$ there exists $g' \in G'$ such that $\psi(A^{(g)}) = A^{(g')}$.

STARTING POINT

Equivalent Gradings:

Definition

We say that gradings $\Gamma : A = \bigoplus_{g\in G} A^{(g)}$ and $\Gamma' : A = \bigoplus_{g'\in G'} A^{(g')}$ on an algebra A, are equivalent, if there exists an automorphism $\psi : A \to A$ of algebras such that for any $g \in \operatorname{supp}\Gamma$[a] there exists $g' \in G'$ such that $\psi(A^{(g)}) = A^{(g')}$. If $\psi = \mathrm{Id}_A$, we say that Γ and Γ' are realizations of the same grading on A as, respectively, a G- and a G'-grading (or that Γ can be regraded by G').

[a] $\operatorname{supp}\Gamma := \{g \in G \mid A^{(g)} \neq 0\}$ is called the support of the grading

STARTING POINT

Definition

Let Γ be a group grading on an algebra A. Suppose that Γ admits a realization as a G_Γ-grading for some group G_Γ and denote by $\varkappa$ the corresponding embedding $\operatorname{supp}\Gamma \hookrightarrow G_\Gamma$. We say that $(G_\Gamma, \varkappa)$ is the universal group of the grading Γ if for any realization of Γ as a grading by a group G with $\psi\colon \operatorname{supp}\Gamma \hookrightarrow G$ there exists a unique group homomorphism $\varphi\colon G_\Gamma \to G$ such that the following diagram is commutative:

$$\begin{array}{ccc} \operatorname{supp}\Gamma & \xrightarrow{\varkappa} & G_\Gamma \\ & \underset{\psi}{\searrow} & \Big\downarrow{\scriptstyle \varphi} \\ & & G \end{array}$$

STARTING POINT

$(G_\Gamma, \varkappa)$ is the initial object of the category $\mathcal{C}_\Gamma$ defined as follows:

STARTING POINT

$(G_\Gamma, \varkappa)$ is the initial object of the category $\mathcal{C}_\Gamma$ defined as follows:

- The objects are all pairs (G, ψ) such that G is a group and Γ can be realized as a G-grading with $\psi\colon \operatorname{supp}\Gamma \hookrightarrow G$ being the embedding of the support;

STARTING POINT

$(G_\Gamma, \varkappa)$ is the initial object of the category $\mathcal{C}_\Gamma$ defined as follows:

- The objects are all pairs (G, ψ) such that G is a group and Γ can be realized as a G-grading with $\psi\colon \operatorname{supp}\Gamma \hookrightarrow G$ being the embedding of the support;

- The morphisms between (G_1, ψ_1) and (G_2, ψ_2) are all group homomorphisms $f\colon G_1 \to G_2$ such that the diagram below is commutative:

$$\begin{array}{ccc} \operatorname{supp}\Gamma & \xrightarrow{\psi_1} & G_1 \\ & \searrow^{\psi_2} & \downarrow f \\ & & G_2 \end{array}$$

PRELIMINARIES

Comodule algebras:

Let H be a Hopf algebra.

PRELIMINARIES

Comodule algebras:

Let H be a Hopf algebra. An algebra A is called a (right) H-comodule algebra if it admits a H-comodule structure $\rho\colon A \to A \otimes H$ which is an algebra homomorphism.

PRELIMINARIES

Comodule algebras:

Let H be a Hopf algebra. An algebra A is called a (right) H-comodule algebra if it admits a H-comodule structure $\rho\colon A \to A \otimes H$ which is an algebra homomorphism.

The map ρ is called a comodule algebra structure on A and will be written in Sweedler notation as $\rho(a) = a_{(0)} \otimes a_{(1)}$, for all $a \in A$.

PRELIMINARIES

Comodule algebras:

Let H be a Hopf algebra. An algebra A is called a (right) H-comodule algebra if it admits a H-comodule structure $\rho\colon A \to A \otimes H$ which is an algebra homomorphism.

The map ρ is called a comodule algebra structure on A and will be written in Sweedler notation as $\rho(a) = a_{(0)} \otimes a_{(1)}$, for all $a \in A$.

Any H-comodule algebra map $\rho\colon A \to A \otimes H$ gives rise to an algebra homomorphism $\zeta\colon H^* \to \mathrm{End}_F(A)$ defined by: $\zeta(h^*)a = h^*(a_{(1)})a_{(0)}$ for all $a \in A$ and $h^* \in H^*$.

PRELIMINARIES

Module algebras:

An algebra A is called a (left) H-module algebra if it admits a H-module structure such that:

$$h(ab) = (h_{(1)}a)(h_{(2)}b), \qquad h\,1_A = \varepsilon(h)\,1_A$$

for all $a, b \in A, h \in H$.

PRELIMINARIES

Module algebras:

An algebra A is called a (left) H-module algebra if it admits a H-module structure such that:

$$h(ab) = (h_{(1)}a)(h_{(2)}b), \qquad h\,1_A = \varepsilon(h)\,1_A$$

for all $a, b \in A, h \in H$.

We denote by ζ the homomorphism of algebras $H \to \mathrm{End}_F(A)$ defined by $\zeta(h)a = ha$, for all $h \in H$ and $a \in A$, and we call it a module algebra structure on A.

NOTATIONAL CONVENTIONS

- In what follows F is a field;

NOTATIONAL CONVENTIONS

- In what follows F is a field;
- $\mathbf{Alg}_F$ (resp. $\mathbf{Coalg}_F$) - the category of (co)unital, (co)associative (co)algebras over F;

NOTATIONAL CONVENTIONS

- In what follows F is a field;
- $\mathbf{Alg}_F$ (resp. $\mathbf{Coalg}_F$) - the category of (co)unital, (co)associative (co)algebras over F;
- $\mathbf{Hopf}_F$ - the category of Hopf algebras;

NOTATIONAL CONVENTIONS

- In what follows F is a field;
- $\mathbf{Alg}_F$ (resp. $\mathbf{Coalg}_F$) - the category of (co)unital, (co)associative (co)algebras over F;
- $\mathbf{Hopf}_F$ - the category of Hopf algebras;
- Let $L\colon \mathbf{Coalg}_F \to \mathbf{Hopf}_F$ be the left adjoint for the forgetful functor $\mathbf{Hopf}_F \to \mathbf{Coalg}_F$ and we denote by $\eta\colon \mathrm{id}_{\mathbf{Coalg}_F} \Rightarrow L$ the unit of this adjunction;

NOTATIONAL CONVENTIONS

- In what follows F is a field;
- $\textbf{Alg}_F$ (resp. $\textbf{Coalg}_F$) - the category of (co)unital, (co)associative (co)algebras over F;
- $\textbf{Hopf}_F$ - the category of Hopf algebras;
- Let $L\colon \textbf{Coalg}_F \to \textbf{Hopf}_F$ be the left adjoint for the forgetful functor $\textbf{Hopf}_F \to \textbf{Coalg}_F$ and we denote by $\eta\colon \mathrm{id}_{\textbf{Coalg}_F} \Rightarrow L$ the unit of this adjunction;
- Let $R\colon \textbf{Alg}_F \to \textbf{Hopf}_F$ be the right adjoint for the forgetful functor $\textbf{Hopf}_F \to \textbf{Alg}_F$.

NOTATIONAL CONVENTIONS

- In what follows F is a field;
- $\mathbf{Alg}_F$ (resp. $\mathbf{Coalg}_F$) - the category of (co)unital, (co)associative (co)algebras over F;
- $\mathbf{Hopf}_F$ - the category of Hopf algebras;
- Let $L\colon \mathbf{Coalg}_F \to \mathbf{Hopf}_F$ be the left adjoint for the forgetful functor $\mathbf{Hopf}_F \to \mathbf{Coalg}_F$ and we denote by $\eta\colon \mathrm{id}_{\mathbf{Coalg}_F} \Rightarrow L$ the unit of this adjunction;
- Let $R\colon \mathbf{Alg}_F \to \mathbf{Hopf}_F$ be the right adjoint for the forgetful functor $\mathbf{Hopf}_F \to \mathbf{Alg}_F$.The counit of this adjunction will be denoted by $\mu\colon R \Rightarrow \mathrm{id}_{\mathbf{Alg}_F}$.

SUPPORT EQUIVALENCE OF COMODULE STRUCTURES

Definition

Let A be an algebra, H_i Hopf algebras, $i = 1, 2$ and assume there exists an H_i-comodule algebra structure on A.

SUPPORT EQUIVALENCE OF COMODULE STRUCTURES

Definition

Let A be an algebra, H_i Hopf algebras, $i = 1, 2$ and assume there exists an H_i-comodule algebra structure on A. The two comodule algebra structures on A are called *support equivalent* if

$$\zeta_1(H_1^*) = \zeta_2(H_2^*)$$

where ζ_i is the algebra homomorphism $H_i^* \to \mathrm{End}_F(A_i)$ induced by the comodule algebra structure on A.

Let A be an H-comodule algebra via $\rho\colon A \to A \otimes H$ and the corresponding homomorphism of algebras $\zeta\colon H^* \to \mathrm{End}_F(A)$.

EQUIVALENCE OF COMODULE ALGEBRA STRUCTURES

SUPPORT

Let A be an H-comodule algebra via $\rho\colon A \to A \otimes H$ and the corresponding homomorphism of algebras $\zeta\colon H^* \to \mathrm{End}_F(A)$.

Choose a basis $(a_\alpha)_\alpha$ in A and let $\rho(a_\alpha) = \sum_\beta a_\beta \otimes h_{\beta\alpha}$ where $h_{\beta\alpha} \in H$.

EQUIVALENCE OF COMODULE ALGEBRA STRUCTURES

SUPPORT

Let A be an H-comodule algebra via $\rho\colon A \to A \otimes H$ and the corresponding homomorphism of algebras $\zeta\colon H^* \to \mathrm{End}_F(A)$.

Choose a basis $(a_\alpha)_\alpha$ in A and let $\rho(a_\alpha) = \sum_\beta a_\beta \otimes h_{\beta\alpha}$ where $h_{\beta\alpha} \in H$.

Let $C(\rho)$ be the F-linear span of all such $h_{\alpha\beta}$;

EQUIVALENCE OF COMODULE ALGEBRA STRUCTURES

SUPPORT

Let A be an H-comodule algebra via $\rho\colon A \to A \otimes H$ and the corresponding homomorphism of algebras $\zeta\colon H^* \to \mathrm{End}_F(A)$.

Choose a basis $(a_\alpha)_\alpha$ in A and let $\rho(a_\alpha) = \sum_\beta a_\beta \otimes h_{\beta\alpha}$ where $h_{\beta\alpha} \in H$.

Let $C(\rho)$ be the F-linear span of all such $h_{\alpha\beta}$; $C(\rho)$ is a subcoalgebra of H:

$$\Delta(h_{\alpha\beta}) = \sum_\gamma h_{\alpha\gamma} \otimes h_{\gamma\beta}, \qquad \varepsilon(h_{\alpha\beta}) = \delta_{\alpha,\beta} \text{ for all } \alpha, \beta.$$

EQUIVALENCE OF COMODULE STRUCTURES

Proposition

Let A be an H_i-comodule algebra for Hopf algebras H_i, $i = 1, 2$. Then the two comodule algebra structures on A are equivalent if and only if there exists an isomorphism of coalgebras $\tau : C(\rho_1) \xrightarrow{\sim} C(\rho_2)$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\rho_1} & A \otimes C(\rho_1) \\ & \searrow^{\rho_2} & \downarrow{\scriptstyle 1_A \otimes \tau} \\ & & A \otimes C(\rho_2) \end{array}$$

THE UNIVERSAL HOPF ALGEBRA

Construction:

THE UNIVERSAL HOPF ALGEBRA

Construction:

- Let $\rho\colon A \to A \otimes H$ be a H-comodule algebra structure on A;

THE UNIVERSAL HOPF ALGEBRA

Construction:

- Let $\rho : A \to A \otimes H$ be a H-comodule algebra structure on A;
- Choose a basis $(a_\alpha)_\alpha$ in A and $h_{\alpha\beta}$ such that $\rho(a_\alpha) = \sum_\beta a_\beta \otimes h_{\beta\alpha}$ and $C(\rho)$ the F-linear span of all such $h_{\alpha\beta}$;

THE UNIVERSAL HOPF ALGEBRA

Construction:

- Let $\rho \colon A \to A \otimes H$ be a H-comodule algebra structure on A;
- Choose a basis $(a_\alpha)_\alpha$ in A and $h_{\alpha\beta}$ such that $\rho(a_\alpha) = \sum_\beta a_\beta \otimes h_{\beta\alpha}$ and $C(\rho)$ the F-linear span of all such $h_{\alpha\beta}$;
- A remains a $C(\rho)$-comodule and therefore a $L(C(\rho))$-comodule.

THE UNIVERSAL HOPF ALGEBRA

Construction:

THE UNIVERSAL HOPF ALGEBRA

Construction:

- Let $a_\alpha a_\beta = \sum_\gamma k^\gamma_{\alpha\beta} a_\gamma$ for some structure constants $k^\gamma_{\alpha\beta} \in F$ and denote by I_0 the ideal of $L(C(\rho))$ generated by:

$$\sum_{r,q} k^\gamma_{rq} \eta_{C(\rho_0)}(h_{r\alpha}) \eta_{C(\rho_0)}(h_{q\beta}) - \sum_u k^u_{\alpha\beta} \eta_{C(\rho_0)}(h_{\gamma u})$$

for all possible choices of indices α, β, γ;

THE UNIVERSAL HOPF ALGEBRA

Construction:

- Let $a_\alpha a_\beta = \sum_\gamma k_{\alpha\beta}^\gamma a_\gamma$ for some structure constants $k_{\alpha\beta}^\gamma \in F$ and denote by I_0 the ideal of $L(C(\rho))$ generated by:

$$\sum_{r,q} k_{rq}^\gamma \eta_{C(\rho_0)}(h_{r\alpha})\eta_{C(\rho_0)}(h_{q\beta}) - \sum_u k_{\alpha\beta}^u \eta_{C(\rho_0)}(h_{\gamma u})$$

  for all possible choices of indices α, β, γ;

- I_0 is a coideal.

THE UNIVERSAL HOPF ALGEBRA

Construction:

THE UNIVERSAL HOPF ALGEBRA

Construction:

- Let I be the ideal generated by spaces $S^n(I_0)$ for all $n \in \mathbb{Z}_+$ where S is the antipode of $L(C(\rho))$.

THE UNIVERSAL HOPF ALGEBRA

Construction:

- Let I be the ideal generated by spaces $S^n(I_0)$ for all $n \in \mathbb{Z}_+$ where S is the antipode of $L(C(\rho))$. Obviously, I is a Hopf ideal and $H^\rho := L(C(\rho))/I$ is a Hopf algebra;

THE UNIVERSAL HOPF ALGEBRA

Construction:

- Let I be the ideal generated by spaces $S^n(I_0)$ for all $n \in \mathbb{Z}_+$ where S is the antipode of $L(C(\rho))$. Obviously, I is a Hopf ideal and $H^\rho := L(C(\rho))/I$ is a Hopf algebra;
- Denote by $\bar{\eta}\colon C(\rho) \to L(C(\rho))/I$ the map induced by $\eta_{C(\rho)}$ and define an H^ρ-comodule algebra structure $\varkappa^\rho$ on A by $\varkappa^\rho(a_\alpha) := \sum_\beta a_\beta \otimes \bar{\eta}(h_{\beta\alpha})$;

THE UNIVERSAL HOPF ALGEBRA

Construction:

- Let I be the ideal generated by spaces $S^n(I_0)$ for all $n \in \mathbb{Z}_+$ where S is the antipode of $L(C(\rho))$. Obviously, I is a Hopf ideal and $H^\rho := L(C(\rho))/I$ is a Hopf algebra;
- Denote by $\bar{\eta}\colon C(\rho) \to L(C(\rho))/I$ the map induced by $\eta_{C(\rho)}$ and define an H^ρ-comodule algebra structure $\varkappa^\rho$ on A by $\varkappa^\rho(a_\alpha) := \sum_\beta a_\beta \otimes \bar{\eta}(h_{\beta\alpha})$;
- We call the pair $(H^\rho, \varkappa^\rho)$ the *universal Hopf algebra* of ρ.

THE UNIVERSAL HOPF ALGEBRA

Consider the category $\mathcal{C}_A^H$ defined as follows:

THE UNIVERSAL HOPF ALGEBRA

Consider the category $\mathcal{C}_A^H$ defined as follows:

1. The objects are H_1-comodule algebra structures on the algebra A equivalent to the given comodule algebra structure (H, ρ)

THE UNIVERSAL HOPF ALGEBRA

Consider the category $\mathcal{C}_A^H$ defined as follows:

1. The objects are H_1-comodule algebra structures on the algebra A equivalent to the given comodule algebra structure (H, ρ) (i.e. H_1-comodule algebra structures on A such that $\zeta_1(H_1^*) = \zeta(H^*)$);

THE UNIVERSAL HOPF ALGEBRA

Consider the category $\mathcal{C}_A^H$ defined as follows:

1. The objects are H_1-comodule algebra structures on the algebra A equivalent to the given comodule algebra structure (H, ρ) (i.e. H_1-comodule algebra structures on A such that $\zeta_1(H_1^*) = \zeta(H^*)$);

2. The morphisms from an H_1-comodule algebra structure on A to an H_2-comodule algebra structure are all Hopf algebra homomorphisms $\tau : H_1 \to H_2$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathrm{End}_F(A) & \xleftarrow{\ \zeta_2\ } & H_2^* \\
 & {\scriptstyle \zeta_1} \nwarrow & \downarrow {\scriptstyle \tau^*} \\
 & & H_1^*
\end{array}
$$

THE UNIVERSAL HOPF ALGEBRA

Theorem

The pair $(H^\rho, \varkappa^\rho)$ is the initial object of the category $\mathcal{C}_A^H$.

THE UNIVERSAL HOPF ALGEBRA

Connection to the universal group of a grading:

THE UNIVERSAL HOPF ALGEBRA

Connection to the universal group of a grading:

Theorem

Let $\Gamma\colon A = \bigoplus_{g \in G} A^{(g)}$ *be a grading on an algebra* A *by a group* G *and consider* $\rho\colon A \to A \otimes FG$ *the corresponding comodule algebra map. If* G_Γ *is the universal group of* Γ *and* $\rho_\Gamma\colon A \to A \otimes FG_\Gamma$ *the corresponding comodule algebra map, then* (FG_Γ, ρ_Γ) *is the universal Hopf algebra of the comodule algebra structure* ρ*.*

THE UNIVERSAL HOPF ALGEBRA

Connection to the universal group of a grading:

Theorem

Let $\Gamma\colon A = \bigoplus_{g\in G} A^{(g)}$ be a grading on an algebra A by a group G and consider $\rho\colon A \to A \otimes FG$ the corresponding comodule algebra map. If G_Γ is the universal group of Γ and $\rho_\Gamma\colon A \to A \otimes FG_\Gamma$ the corresponding comodule algebra map, then (FG_Γ, ρ_Γ) is the universal Hopf algebra of the comodule algebra structure ρ.

THE UNIVERSAL HOPF ALGEBRA AS A FUNCTOR

Given an algebra A, we define the category $\mathcal{C}_A$ as follows:

THE UNIVERSAL HOPF ALGEBRA AS A FUNCTOR

Given an algebra A, we define the category $\mathcal{C}_A$ as follows:

1. The objects are pairs (H, ρ) where H is a Hopf algebra and $\rho\colon A \to A \otimes H$ is a right H-comodule algebra structure on A;

THE UNIVERSAL HOPF ALGEBRA AS A FUNCTOR

Given an algebra A, we define the category $\mathcal{C}_A$ as follows:

1. The objects are pairs (H, ρ) where H is a Hopf algebra and $\rho\colon A \to A \otimes H$ is a right H-comodule algebra structure on A;

2. The morphisms between two objects (H_1, ρ_1) and (H_2, ρ_2) are coalgebra homomorphisms $\tau\colon C(\rho_1) \to C(\rho_2)$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\rho_1} & A \otimes C(\rho_1) \\ & \searrow^{\rho_2} & \downarrow{\scriptstyle \mathrm{id}_A \otimes \tau} \\ & & A \otimes C(\rho_2) \end{array}$$

THE UNIVERSAL HOPF ALGEBRA AS A FUNCTOR

THE UNIVERSAL HOPF ALGEBRA AS A FUNCTOR

Any morphism $\tau\colon (H_1,\ \rho_1) \to (H_2,\ \rho_2)$ in $\mathcal{C}_A$ induces a Hopf algebra homomorphism $\overline{\tau}$ between the corresponding universal Hopf algebras H^{ρ_1} and H^{ρ_2}, respectively.

THE UNIVERSAL HOPF ALGEBRA AS A FUNCTOR

Any morphism $\tau\colon (H_1,\ \rho_1) \to (H_2,\ \rho_2)$ in $\mathcal{C}_A$ induces a Hopf algebra homomorphism $\overline{\tau}$ between the corresponding universal Hopf algebras H^{ρ_1} and H^{ρ_2}, respectively.

Theorem

*There exists a functor $F\colon \mathcal{C}_A \to$ **Hopf**$_F$ given as follows:*

$$F(H, \rho) = H^{\rho} \text{ and } F(\tau) = \overline{\tau}.$$

THE UNIVERSAL HOPF ALGEBRA

Hopf-Galois extensions:

THE UNIVERSAL HOPF ALGEBRA

Hopf-Galois extensions: Let $\rho\colon A \to A \otimes H$ be a comodule algebra structure on A and $A^{\mathrm{co}H} := \{a \in A \mid \rho(a) = a \otimes 1_H\}$.

THE UNIVERSAL HOPF ALGEBRA

Hopf-Galois extensions: Let $\rho\colon A \to A \otimes H$ be a comodule algebra structure on A and $A^{\mathrm{co}H} := \{a \in A \mid \rho(a) = a \otimes 1_H\}$. A is called a *Hopf–Galois extension* of $A^{\mathrm{co}H}$ if the linear map $\mathrm{can}\colon A \otimes_{A^{\mathrm{co}H}} A \to A \otimes H$ defined below is bijective:

$$\mathrm{can}(a \otimes b) := ab_{(0)} \otimes b_{(1)}.$$

THE UNIVERSAL HOPF ALGEBRA

Hopf-Galois extensions: Let $\rho\colon A \to A \otimes H$ be a comodule algebra structure on A and $A^{\mathrm{co}H} := \{a \in A \mid \rho(a) = a \otimes 1_H\}$. A is called a *Hopf–Galois extension* of $A^{\mathrm{co}H}$ if the linear map $\mathrm{can}\colon A \otimes_{A^{\mathrm{co}H}} A \to A \otimes H$ defined below is bijective:

$$\mathrm{can}(a \otimes b) := ab_{(0)} \otimes b_{(1)}.$$

Theorem

If $A/A^{\mathrm{co}H}$ is a Hopf–Galois extension then (H, ρ) is the universal Hopf algebra of ρ.

THE UNIVERSAL HOPF ALGEBRA

Corollary

Let H be a Hopf algebra. Then the universal Hopf algebra of the H-comodule algebra structure on H defined by the comultiplication $\Delta\colon H \to H \otimes H$ is again (H, Δ).

THE UNIVERSAL HOPF ALGEBRA

Example

Let H be a Hopf algebra and A an H-module algebra. We denote by $A \# H$ the corresponding *smash product*,

THE UNIVERSAL HOPF ALGEBRA

Example

Let H be a Hopf algebra and A an H-module algebra. We denote by $A\#H$ the corresponding *smash product*, i.e. $A\#H = A \otimes H$ as a vector space with multiplication given as follows:

$$(a\#h)(b\#g) = a(h_{(1)} \cdot b)\#h_{(2)}g$$

where we denote the element $a \otimes h \in A \otimes H$ by $a\#h$. We have an H-comodule algebra structure on $A\#H$ given by:

$$\rho\colon A\#H \to (A\#H) \otimes H, \quad \rho(a\#h) = a\#h_{(1)} \otimes h_{(2)}$$

THE UNIVERSAL HOPF ALGEBRA

Example

Let H be a Hopf algebra and A an H-module algebra. We denote by $A\#H$ the corresponding *smash product*, i.e. $A\#H = A \otimes H$ as a vector space with multiplication given as follows:

$$(a\#h)(b\#g) = a(h_{(1)} \cdot b)\#h_{(2)}g$$

where we denote the element $a \otimes h \in A \otimes H$ by $a\#h$. We have an H-comodule algebra structure on $A\#H$ given by:

$$\rho\colon A\#H \to (A\#H) \otimes H, \quad \rho(a\#h) = a\#h_{(1)} \otimes h_{(2)}$$

The universal Hopf algebra of ρ is $(H, \mathrm{id}_A \otimes \Delta)$.

EQUIVALENCE OF MODULE STRUCTURES

Definition

Let A be an algebra, H_i Hopf algebras, $i = 1, 2$ and assume there exists an H_i-module algebra structure on A.

EQUIVALENCE OF MODULE STRUCTURES

Definition

Let A be an algebra, H_i Hopf algebras, $i = 1, 2$ and assume there exists an H_i-module algebra structure on A. The two module algebra structures on A are called *equivalent* if

$$\zeta_1(H_1) = \zeta_2(H_2)$$

where ζ_i is the module algebra structure on A.

EQUIVALENCE OF MODULE STRUCTURES

Proposition

Let $\rho_i \colon A \to A \otimes H_i$*,* $i = 1, 2$*, be two comodule algebra structures on an algebra* A *where* H_i *are finite dimensional Hopf algebras and denote by* $\zeta_i \colon H_i^* \to \mathrm{End}_F(A)$ *the corresponding homomorphisms of algebras.*

EQUIVALENCE OF MODULE STRUCTURES

Proposition

Let $\rho_i\colon A \to A \otimes H_i$, $i = 1, 2$, be two comodule algebra structures on an algebra A where H_i are finite dimensional Hopf algebras and denote by $\zeta_i\colon H_i^* \to \mathrm{End}_F(A)$ the corresponding homomorphisms of algebras. Then ρ_1 and ρ_2 are equivalent comodule algebra structures if and only if ζ_1 and ζ_2 are equivalent module algebra structures.

RATIONAL ACTIONS OF AFFINE ALGEBRAIC GROUPS

Let G be an affine algebraic group over an algebraically closed field F.

RATIONAL ACTIONS OF AFFINE ALGEBRAIC GROUPS

Let G be an affine algebraic group over an algebraically closed field F.

Then $\mathcal{O}(G)$ the algebra of regular functions on G is a Hopf algebra.

RATIONAL ACTIONS OF AFFINE ALGEBRAIC GROUPS

Let G be an affine algebraic group over an algebraically closed field F.

Then $\mathcal{O}(G)$ the algebra of regular functions on G is a Hopf algebra.

Suppose G is acting *rationally* by automorphisms on a finite dimensional algebra A,

RATIONAL ACTIONS OF AFFINE ALGEBRAIC GROUPS

Let G be an affine algebraic group over an algebraically closed field F.

Then $\mathcal{O}(G)$ the algebra of regular functions on G is a Hopf algebra.

Suppose G is acting *rationally* by automorphisms on a finite dimensional algebra A, i.e. for a given basis $a_1, \ldots, a_n$ in A there exist $\omega_{ij} \in \mathcal{O}(G)$, where $1 \leqslant i, j \leqslant n$, such that $ga_j = \sum_{i=1}^{n} \omega_{ij}(g) a_i$ for all $1 \leqslant j \leqslant n$ and $g \in G$.

RATIONAL ACTIONS OF AFFINE ALGEBRAIC GROUPS

Let G be an affine algebraic group over an algebraically closed field F.

Then $\mathcal{O}(G)$ the algebra of regular functions on G is a Hopf algebra.

Suppose G is acting *rationally* by automorphisms on a finite dimensional algebra A, i.e. for a given basis $a_1, \ldots, a_n$ in A there exist $\omega_{ij} \in \mathcal{O}(G)$, where $1 \leqslant i, j \leqslant n$, such that $ga_j = \sum_{i=1}^{n} \omega_{ij}(g)a_i$ for all $1 \leqslant j \leqslant n$ and $g \in G$.

1. A is an $\mathcal{O}(G)$-comodule algebra where: $\rho(a_j) := \sum_{i=1}^{n} a_i \otimes \omega_{ij}$ for $1 \leqslant j \leqslant n$.

RATIONAL ACTIONS OF AFFINE ALGEBRAIC GROUPS

Let G be an affine algebraic group over an algebraically closed field F.

Then $\mathcal{O}(G)$ the algebra of regular functions on G is a Hopf algebra.

Suppose G is acting *rationally* by automorphisms on a finite dimensional algebra A, i.e. for a given basis $a_1, \ldots, a_n$ in A there exist $\omega_{ij} \in \mathcal{O}(G)$, where $1 \leqslant i, j \leqslant n$, such that $ga_j = \sum_{i=1}^{n} \omega_{ij}(g)a_i$ for all $1 \leqslant j \leqslant n$ and $g \in G$.

1. A is an $\mathcal{O}(G)$-comodule algebra where:
   $\rho(a_j) := \sum_{i=1}^{n} a_i \otimes \omega_{ij}$ for $1 \leqslant j \leqslant n$.
2. A is an $\mathcal{O}(G)^{\circ}$-module algebra:
   $f^* a_j = \sum_{i=1}^{n} f^*(\omega_{ij})a_i$ for all $1 \leqslant j \leqslant n$ and $f^* \in O(G)^{\circ}$.

1. The Lie algebra $\mathfrak{g}$ of G is the subspace consisting of all *primitive* elements of $\mathcal{O}(G)^\circ$, i.e. $f^* \in \mathcal{O}(G)^\circ$ such that $\Delta(f^*) = f^* \otimes 1 + 1 \otimes f^*$, and the $\mathfrak{g}$-action on A by derivations is just the restriction of the $\mathcal{O}(G)^\circ$-action

1. The Lie algebra $\mathfrak{g}$ of G is the subspace consisting of all *primitive* elements of $\mathcal{O}(G)^\circ$, i.e. $f^* \in \mathcal{O}(G)^\circ$ such that $\Delta(f^*) = f^* \otimes 1 + 1 \otimes f^*$, and the $\mathfrak{g}$-action on A by derivations is just the restriction of the $\mathcal{O}(G)^\circ$-action
2. The group G itself can be identified with the group of *group-like* elements of $\mathcal{O}(G)^\circ$, i.e. $f^* \in \mathcal{O}(G)^\circ$ such that $\Delta(f^*) = f^* \otimes f^*$.

RATIONAL ACTIONS OF AFFINE ALGEBRAIC GROUPS

Three Hopf algebras are acting on A: $\mathcal{O}(G)^\circ$, FG and $U(\mathfrak{g})$.

RATIONAL ACTIONS OF AFFINE ALGEBRAIC GROUPS

Three Hopf algebras are acting on A: $\mathcal{O}(G)^{\circ}$, FG and $U(\mathfrak{g})$.

Theorem

Let G be a connected affine algebraic group over an algebraically closed field F of characteristic 0 acting rationally by automorphisms on a finite dimensional algebra A. Let $\mathfrak{g}$ be the Lie algebra of G. Then the corresponding FG-action, $U(\mathfrak{g})$-action and $\mathcal{O}(G)^{\circ}$-action on A are equivalent.

THE UNIVERSAL HOPF ALGEBRA

Construction:

THE UNIVERSAL HOPF ALGEBRA

Construction:

- Let $\zeta : H \to \mathrm{End}_F(A)$ be an H-module algebra on A ;

THE UNIVERSAL HOPF ALGEBRA

Construction:

- Let $\zeta \colon H \to \mathrm{End}_F(A)$ be an H-module algebra on A ;
- For any H_1-module algebra $\zeta_1 \colon H_1 \to \mathrm{End}_F(A)$ on A that is equivalent to ζ there exists a unique Hopf algebra homomorphism $\varphi_1 \colon H_1 \to R(\zeta(H))$ such that the following diagram commutes:

$$\begin{array}{ccc} H_1 & \xrightarrow{\varphi_1} & R(\zeta(H)) \\ & \searrow^{\zeta_1} & \downarrow{\tilde{\zeta}} \\ & & \zeta(H) \end{array}$$

where we denote $\tilde{\zeta} = \mu_{\zeta(H)}$.

THE UNIVERSAL HOPF ALGEBRA OF A MODULE ALGEBRA

Construction:

THE UNIVERSAL HOPF ALGEBRA OF A MODULE ALGEBRA

Construction:

- Consider the subalgebra H_ζ of $R(\zeta(H))$ generated by $\varphi_1(H_1)$ for all such structures ζ_1;

THE UNIVERSAL HOPF ALGEBRA OF A MODULE ALGEBRA

Construction:

- Consider the subalgebra H_ζ of $R(\zeta(H))$ generated by $\varphi_1(H_1)$ for all such structures ζ_1;
- H_ζ is a Hopf algebra and $\psi_\zeta := \tilde{\zeta}|_{H_\zeta}$ defines a H_ζ-module algebra structure on A;

THE UNIVERSAL HOPF ALGEBRA OF A MODULE ALGEBRA

Construction:

- Consider the subalgebra H_ζ of $R(\zeta(H))$ generated by $\varphi_1(H_1)$ for all such structures ζ_1;
- H_ζ is a Hopf algebra and $\psi_\zeta := \tilde{\zeta}|_{H_\zeta}$ defines a H_ζ-module algebra structure on A;
- We call (H_ζ, ψ_ζ) the *universal Hopf algebra* of ζ.

THE UNIVERSAL HOPF ALGEBRA

Consider the category ${}_H\mathcal{C}_A$ defined as follows:

THE UNIVERSAL HOPF ALGEBRA

Consider the category ${}_H\mathcal{C}_A$ defined as follows:

1. The objects are H_1-module algebra structures on the algebra A equivalent to the given module algebra structure

THE UNIVERSAL HOPF ALGEBRA

Consider the category ${}_H\mathcal{C}_A$ defined as follows:

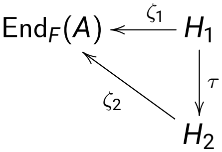
1. The objects are H_1-module algebra structures on the algebra A equivalent to the given module algebra structure (i.e. H_1-module algebra structures on A such that $\zeta_1(H_1) = \zeta(H)$);

THE UNIVERSAL HOPF ALGEBRA

Consider the category ${}_H\mathcal{C}_A$ defined as follows:

1. The objects are H_1-module algebra structures on the algebra A equivalent to the given module algebra structure (i.e. H_1-module algebra structures on A such that $\zeta_1(H_1) = \zeta(H)$);

2. The morphisms from an H_1-module algebra structure on A to an H_2-module algebra structure are all Hopf algebra homomorphisms $\tau : H_1 \to H_2$ such that the following diagram is commutative:



THE UNIVERSAL HOPF ALGEBRA

Theorem

The pair (H_ζ, ψ_ζ) is the final object of the category ${}_H\mathcal{C}_A$.

THE UNIVERSAL HOPF ALGEBRA AS A FUNCTOR

Given an algebra A, we define the category ${}_A\mathcal{C}$ as follows:

THE UNIVERSAL HOPF ALGEBRA AS A FUNCTOR

Given an algebra A, we define the category ${}_A\mathcal{C}$ as follows:

1. The objects are pairs (H, ζ) where H is a Hopf algebra and $\zeta\colon H \to \mathrm{End}_F(A)$ is a left H-module algebra structure on A;

THE UNIVERSAL HOPF ALGEBRA AS A FUNCTOR

Given an algebra A, we define the category ${}_A\mathcal{C}$ as follows:

1. The objects are pairs (H, ζ) where H is a Hopf algebra and $\zeta \colon H \to \mathrm{End}_F(A)$ is a left H-module algebra structure on A;
2. The morphisms between two objects (H, ζ) and $(H^{'}, \zeta^{'})$ are algebra homomorphisms $\lambda : H/\ker\zeta \to H^{'}/\ker\zeta^{'}$ such that the following diagram is commutative:

$$\begin{array}{ccc} H/\ker\zeta & \xrightarrow{\tilde{\zeta}} & \mathrm{End}_F(A) \\ {\scriptstyle\lambda}\downarrow & \nearrow_{\tilde{\zeta}'} & \\ H^{'}/\ker\zeta^{'} & & \end{array}$$

where $\tilde{\zeta} \colon H/\ker\zeta \to \mathrm{End}_F(A)$ (resp. $\tilde{\zeta}' : H^{'}/\ker\zeta^{'} \to \mathrm{End}_F(A)$) are induced by ζ (resp. $\zeta^{'}$)), i.e. $\tilde{\zeta}(\tilde{x}) = \zeta(x)$ for all $\tilde{x} \in H/\ker\zeta$.

THE UNIVERSAL HOPF ALGEBRA AS A FUNCTOR

THE UNIVERSAL HOPF ALGEBRA AS A FUNCTOR

1. Any morphism $\lambda : H/\ker \zeta \to H^{'}/\ker \zeta^{'}$ in ${}_{A}\mathcal{C}$ induces a Hopf algebra homomorphism $\overline{\lambda}$ between $R(\zeta(H))$ and $R(\zeta^{'}(H^{'}))$, respectively;

THE UNIVERSAL HOPF ALGEBRA AS A FUNCTOR

1. Any morphism $\lambda : H/\ker\zeta \to H^{'}/\ker\zeta^{'}$ in ${}_{A}\mathcal{C}$ induces a Hopf algebra homomorphism $\overline{\lambda}$ between $R(\zeta(H))$ and $R(\zeta^{'}(H^{'}))$, respectively;
2. $\overline{\lambda}_{|H_{\zeta}} \subseteq H^{'}_{\zeta^{'}}$.

THE UNIVERSAL HOPF ALGEBRA AS A FUNCTOR

1. Any morphism $\lambda : H/\ker\zeta \to H^{'}/\ker\zeta^{'}$ in ${}_A\mathcal{C}$ induces a Hopf algebra homomorphism $\overline{\lambda}$ between $R(\zeta(H))$ and $R(\zeta^{'}(H^{'}))$, respectively;
2. $\overline{\lambda}_{|H_\zeta} \subseteq H^{'}_{\zeta^{'}}$.

Theorem

*There exists a functor $G : {}_A\mathcal{C} \to$ **Hopf**$_F$ given as follows:*

$$G(H, \zeta) = H_\zeta \text{ and } G(\lambda) = \overline{\lambda}_{|H_\zeta}.$$

THE UNIVERSAL HOPF ALGEBRA OF A MODULE ALGEBRA

Example

THE UNIVERSAL HOPF ALGEBRA OF A MODULE ALGEBRA

Example

Let H be a Hopf algebra and denote by $\zeta\colon H \to \mathrm{End}_F(H^*)$ the homomorphism defined as follows for all $h, t \in H$, $\lambda \in H^*$:

$$(\zeta(h)\lambda)(t) := \lambda(th).$$

THE UNIVERSAL HOPF ALGEBRA OF A MODULE ALGEBRA

Example

Let H be a Hopf algebra and denote by $\zeta\colon H \to \mathrm{End}_F(H^*)$ the homomorphism defined as follows for all $h, t \in H$, $\lambda \in H^*$:

$$(\zeta(h)\lambda)(t) := \lambda(th).$$

Then ζ is a H-module algebra structure on the algebra H^* and the universal Hopf algebra of ζ is again (H, ζ).

Example

Let F is an algebraically closed field, $\operatorname{char} F = 0$ and $A = F[x]/(x^2)$.

Example

Let F is an algebraically closed field, $\operatorname{char} F = 0$ and $A = F[x]/(x^2)$. Consider G to be the cyclic group of order 2 with the generator c.

Example

Let F is an algebraically closed field, $\operatorname{char} F = 0$ and $A = F[x]/(x^2)$. Consider G to be the cyclic group of order 2 with the generator c. Define a G-action on A by $c\bar{x} = -\bar{x}$.

Example

Let F is an algebraically closed field, $\operatorname{char} F = 0$ and $A = F[x]/(x^2)$. Consider G to be the cyclic group of order 2 with the generator c. Define a G-action on A by $c\bar{x} = -\bar{x}$.

Then the universal Hopf algebra of the corresponding FG-action $\zeta_0 : FG \to \operatorname{End}_F(A)$ equals (H, ζ) where $H = F[y] \otimes FF^{\times}$, where the coalgebra structure on $F[y]$ is defined by:

$$\Delta(y) = 1 \otimes y + y \otimes 1 \text{ and } \varepsilon(y) = 0.$$

Example

Let F is an algebraically closed field, $\mathrm{char}\, F = 0$ and $A = F[x]/(x^2)$. Consider G to be the cyclic group of order 2 with the generator c. Define a G-action on A by $c\bar{x} = -\bar{x}$.

Then the universal Hopf algebra of the corresponding FG-action $\zeta_0 \colon FG \to \mathrm{End}_F(A)$ equals (H, ζ) where $H = F[y] \otimes FF^\times$, where the coalgebra structure on $F[y]$ is defined by:

$$\Delta(y) = 1 \otimes y + y \otimes 1 \text{ and } \varepsilon(y) = 0.$$

The antipode S of H and the action $\zeta \colon H \to \mathrm{End}_F(A)$ are defined by

$$S(y^k \otimes \lambda) = (-1)^k y^k \otimes \lambda^{-1} \text{ and } \zeta(y^k \otimes \lambda)\bar{x} = \lambda$$

for $k \in \mathbb{Z}_+$ and $\lambda \in F^\times$.

Thank you!